# A Truly Concurrent Game Model of the Asynchronous $\pi$-Calculus

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**Abstract.** In game semantics, a computation is represented by a *play*, which is traditionally a sequence of messages exchanged by a program and an environment. Because of the sequentiality of plays, most game models for concurrent programs are a kind of interleaving semantics. Several frameworks for truly concurrent game models have been proposed, but no model has yet been applied to give a semantics of a complex concurrent calculus such as the $\pi$-calculus (with replication).
This paper proposes a truly concurrent version of the HO/N game model in which a play is not a sequence but a directed acyclic graph (DAG) with two kinds of edges, justification pointers and causal edges. By using this model, we give the first truly concurrent game semantics for the asynchronous $\pi$-calculus. In order to illustrate a possible application, we propose an intersection type system for the asynchronous $\pi$-calculus by means of our game model, and discuss when a process can be completely characterised by the intersection type system.

**Keywords:** HO/N game model, true concurrency, asynchronous $\pi$-calculus

## 1 Introduction

Game semantics succeeded to give semantics for variety of programming languages such as PCF [1, 21] and Idealized Algol [2].

The idea of game semantics has been applied to give models for concurrent calculi such as CSP [23], Idealized Parallel Algol [18] and the asynchronous $\pi$-calculus [24]. However, the sequential nature of plays forces these models to be a kind of interleaving semantics; the causalities between events are obfuscated.

Hence it is natural to investigate a concurrent extension of the traditional game models. Several frameworks for concurrent game models have been proposed by several researchers [3, 27, 29, 34], but no model has yet been applied to give a semantics of a complex concurrent calculus such as the $\pi$-calculus (with replication), as pointed out in [10]. The goal of this paper is to develop a truly concurrent game model by which the asynchronous $\pi$-calculus can be interpreted.

The starting point of our development is an observation by Melliès [27]: in the HO/N innocent game model [21, 32], only a part of the sequential information is really relevant. For example, the order of consecutive occurrences of O- and

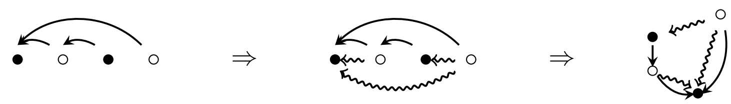


Fig. 1: Idea of the desequentialization.

P-moves are indispensable, whereas that of consecutive occurrences of P- and O-moves can be safely forgotten (unless the O-move is justified by the P-move).

Now it is natural to think of a play in which the relevant order information is made explicit. Consider a traditional sequential play on the left side in Figure 1, where $\bullet$ (resp. $\circ$) represents an O-move (resp. a P-move) and a pointer is a justification pointer. By making the relevant sequential information explicit, we obtain a representation in the middle in Figure 1. Then because all the relevant sequential information has been explicitly indicated by edges, we can simply forget the sequential information, resulting in the right representation in Figure 1. This is our representation of a play that we call a *DAG-based play.*

A DAG-based play generated by this way from a sequential play satisfies a certain property, which reflects the sequential nature of the target language of the innocent game model [21]. In order to model a concurrent calculus, the condition required for DAG-based plays should be weakened. This is the idea that leads us to the definition of plays in this paper.

Following this idea, we develop a DAG-based game model for the asynchronous $\pi$-calculus, guided by the sequential game model of Laird [24]. Our model is truly concurrent in the sense that it distinguishes between $a.\bar{b} \mid c.\bar{d}$ and $a.(\bar{b} \mid c.\bar{d}) + c.(a.\bar{b} \mid \bar{d})$. Laird's model can be reconstructed by lining up the nodes of DAG-based plays of our model. We prove the soundness of our model by reducing it to that of Laird's model, using this relationship.

As a possible application of our model, we give an intersection type system based on the relationship between intersection types and game semantics which has been studied in the case of $\lambda$-calculus [7, 14, 37]. Based on a game-semantic consideration, we characterise a class of processes that are completely described by the intersection type system.

*Organisation of the paper* Section 2 defines our target language, a variant of the asynchronous $\pi$-calculus. In Section 3, we define our truly concurrent game model and relate it with sequential game models. A semantics of the $\pi$-calculus is given in Section 4. Section 5 illustrates a possible application of our game model, giving an intersection type system for a fragment of the $\pi$-calculus. Section 6 discusses related work and Section 7 concludes the paper.

## 2 Simply-Typed Asynchronous $\pi$-Calculus

We define the target language of the paper: the simply-typed asynchronous polyadic $\pi$-calculus with distinction between input and output channels. This

$$\frac{}{\Gamma \vdash \mathbf{0}; \Sigma} \qquad \frac{\Gamma, \bar{x} : T \vdash P; \Sigma, y : T}{\Gamma \vdash \nu(\bar{x}, y).P; \Sigma} \qquad \frac{}{\Gamma, \bar{x} : \mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}], \bar{\boldsymbol{y}} : \boldsymbol{S} \vdash \bar{x}\langle \bar{\boldsymbol{y}}, \boldsymbol{z}\rangle; \Sigma, \boldsymbol{z} : \boldsymbol{T}}$$

$$\frac{\Gamma \vdash P; \Sigma \quad \Gamma' \vdash Q; \Sigma'}{\Gamma, \Gamma' \vdash P|Q; \Sigma, \Sigma'} \quad \frac{\Gamma, \bar{\boldsymbol{y}} : \boldsymbol{S} \vdash P; \Sigma, \boldsymbol{z} : \boldsymbol{T}}{\Gamma \vdash x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P; \Sigma, x{:}\,\mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}]} \quad \frac{\Gamma, \bar{\boldsymbol{y}} : \boldsymbol{S} \vdash P; \Sigma, \boldsymbol{z} : \boldsymbol{T}}{\Gamma \vdash !x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P; \Sigma, x{:}\,\mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}]}$$

Fig. 2: Typing rules. (Contraction and exchange rules are omitted.)

is the calculus studied in the previous work of Laird [24], in which he gave an interleaving (or sequential) game model.

We assume countably infinite sets of input names and of output names. Unlike the standard $\pi$-calculus in which an input name $a$ is *a priori* connected to the output name $\bar{a}$, we do not assume any relationship between input and output names but a connection is established by $\nu$ constructor. This design choice significantly simplifies the denotational semantics.

The *processes* are defined by the following grammar: $P, Q ::= \mathbf{0} \mid \bar{x}\langle \bar{\boldsymbol{y}}, \boldsymbol{z}\rangle \mid x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P \mid P|Q \mid !x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P \mid \nu(\bar{x}, y).P$. Here $x$ (resp. $\bar{x}$) ranges over input (resp. output) names and $\boldsymbol{x}$ (resp. $\bar{\boldsymbol{x}}$) represents a (possibly empty) sequence of input (resp. output) names. Name creation $\nu$ creates a pair of input and output names. We abbreviate $\nu(\bar{x}_1, y_1).\ldots.\nu(\bar{x}_n, y_n).P$ as $\nu(\bar{x}_1 \ldots \bar{x}_n, y_1 \ldots y_n).P$.

The structural congruence $\equiv$ is defined as usual. The *one-step reduction relation* $\longrightarrow$ on processes is defined by the following rule:

$$\nu(\bar{\boldsymbol{z}}, \boldsymbol{w}).\nu(\bar{x}, y).(y(\bar{\boldsymbol{a}}, \boldsymbol{b}).P \mid \bar{x}\langle \bar{\boldsymbol{c}}, \boldsymbol{d}\rangle \mid Q) \longrightarrow \nu(\bar{\boldsymbol{z}}, \boldsymbol{w}).\nu(\bar{x}, y).(P\{\bar{\boldsymbol{c}}/\bar{\boldsymbol{a}}, \boldsymbol{d}/\boldsymbol{b}\} \mid Q)$$

It is worth emphasising here that the communication only occurs over names that are bound by $\nu$. The *reduction relation* $\longrightarrow^*$ is the reflexive transitive closure of $(\longrightarrow \cup \equiv)$. We write $P{\Downarrow_{\bar{x}}}$ if $P \longrightarrow^* \nu(\bar{\boldsymbol{y}}, \boldsymbol{z}).(\bar{x}\langle \bar{\boldsymbol{y}}', \boldsymbol{z}'\rangle \mid Q)$ for some $Q$, where $\bar{x}$ is free. Note that we can observe only an output action.

We require that processes are well-typed. The syntax of *types* is given by $S, T ::= \mathbf{ch}[S_1 \ldots S_m, T_1 \ldots T_n]$. We write $x : \mathbf{ch}[S_1 \ldots S_m, T_1 \ldots T_n]$ to mean that $x$ is an input name by which one receives $m$ output names and $n$ input names at once. Similarly for $\bar{y} : \mathbf{ch}[S_1 \ldots S_m, T_1 \ldots T_n]$. A sequence $S_1 \ldots S_m$ of types is often written as $\boldsymbol{S}$ and the empty sequence is written as $\_$. The type $\mathbf{ch}[\_, \_]$ is abbreviated as $\mathbf{ch}[]$. An *input type environment* is a finite sequence of type bindings of the form $x : T$ and an *output type environment* is that of the form $\bar{y} : S$. A *type judgement* is of the form $\Gamma \vdash P; \Sigma$, where $\Gamma$ and $\Sigma$ are input and output type environments, respectively. Typing rules are listed in Figure 2.

*Remark 1.* (1) A calculus with *a priori* connection between an input name $x$ and an output name $\bar{x}$ can be simulated by passing/receiving a pair $(x, \bar{x})$ of input and output names. Via this translation our game semantics is applicable to a calculus with *a priori* connection because the translation reflects may-testing equivalence. (2) The standard parallel composition, which invokes communications of the two processes, can be expressed as $\nu(\bar{\boldsymbol{a}}\bar{\boldsymbol{b}}, \boldsymbol{a}\boldsymbol{b}).(P|Q|(\boldsymbol{a}' \to \bar{\boldsymbol{a}})|(\boldsymbol{b} \to \bar{\boldsymbol{b}}'))$ where $\boldsymbol{a}$ and $\bar{\boldsymbol{b}}$ are free names in $P$ and $Q$, and $\boldsymbol{a}' \to \bar{\boldsymbol{a}}$ is a "forwarder", a process forwarding names received from $a_i'$ to $\bar{a}_i$.

## 3 Concurrent HO/N Game Model

This section introduces a truly concurrent game model in which a play is not a sequence but a directed acyclic graph (DAG). A node of a play is labelled by a move representing an event; an edge represents either a justification pointer or causality. The key is the notion of plays (Section 3.2) and of interactions (Section 3.3). The other parts are relatively straightforward adaptation of the techniques in the standard HO/N game model (e.g. [21]) or Laird's model [24].

### 3.1 Arenas

The definition of arenas is (essentially) the same as the definition of arenas in the case of the sequential game model of $\pi$-calculus [24]. The differences from the standard definition (e.g. [21]) are (1) all moves are questions, and (2) the owner of moves does not have to alternate.

**Definition 1 (Arena).** *An* arena *is a triple $A = (\mathcal{M}_A, \lambda_A, \vdash_A)$, where $\mathcal{M}_A$ is a set of* moves*, $\lambda_A \colon \mathcal{M}_A \to \{P, O\}$ is an* ownership function *and $\vdash_A \subseteq (\{\star\} + \mathcal{M}_A) \times \mathcal{M}_A$ is an* enabling relation *that satisfies: for every $m \in \mathcal{M}_A$, there uniquely exists $x \in \{\star\} + \mathcal{M}_A$ such that $x \vdash_A m$.*

We say that $m$ is a *P-move* if $\lambda_A(m) = P$; it is an *O-move* if $\lambda_A(m) = O$. Every move represents an output action: a P-move is an output action of the process and an O-move is that of the environment (see a discussion after Definition 4). Let $\lambda_A^{\perp}$ denote the negation of $\lambda_A$ i.e. $\lambda_A^{\perp}(m) = O$ (resp. $\lambda_A^{\perp}(m) = P$) if $\lambda_A(m) = P$ (resp. $\lambda_A(m) = O$). A move $m$ is *initial* if $\star \vdash_A m$. An arena is *negative* (resp. *positive*) if all initial moves are O-moves (resp. P-moves). In what follows, we shall consider only negative arenas (hence we often use arenas to mean negative arenas). The *empty arena* is defined by $I := (\emptyset, \emptyset, \emptyset)$.

Negative and positive arenas correspond to input and output type environments, respectively. Hence a judgement, which consists of a pair of input and output type environments, should be expressed as a pair of arenas.

**Definition 2 (Arena pair).** *An* arena pair *is a pair $(A, B)$ of (negative) arenas. We write $\mathcal{M}_{A,B}$ for $\mathcal{M}_A + \mathcal{M}_B$. The ownership function is defined by $\lambda_{A,B} = [\lambda_A^{\perp}, \lambda_B]$. The enabling relation $\vdash_{A,B}$ is given by: $m \vdash_{A,B} m'$ if and only if $m \vdash_A m'$ or $m \vdash_B m'$. (In particular, $\star \vdash_{A,B} m$ iff $\star \vdash_A m$ or $\star \vdash_B m'$.)*

Note that an arena pair is not a negative arena since it has an initial P-move.

*Example 1.* Three (negative) arenas $A$, $B$ and $C$ are illustrated in Fig. 3, as well as the arena pair $(A, B)$. Those arenas are used in examples in this paper. Nodes are labelled by moves and edges represent the enabling relation. If a name is overlined, the move is a P-move; otherwise it is an O-move. The arena pair $(A, B)$ corresponds to the pair of the output type environment $\varGamma = \bar{a}_1 : \mathbf{ch}[\_, \mathbf{ch}[]], \bar{a}_2 : \mathbf{ch}[]$ and the input type environment $\varSigma = b_1 : \mathbf{ch}[\mathbf{ch}[], \mathbf{ch}[\mathbf{ch}[], \_]]$. (Channel names do not have to coincide with move names.)

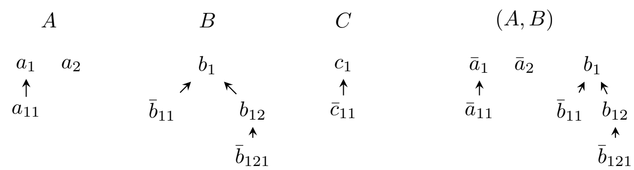



Fig. 3: Examples of arenas and an arena pair.

### 3.2 DAG-based plays

In the standard HO/N game model [21], a play is a sequence of moves equipped with pointers, called *justification pointers*. The justification pointers express the binder-bindee relation and the sequential structure expresses the temporal relation between the events in the sequence (e.g. in the sequence $s_1\, a\, s_2\, b\, s_3$, the event $b$ occurs after $a$). The causal relation is left implicit (cf. Section 3.5).

In the proposed game model, we explicitly describe the causal relation as well as the justification pointers.

**Definition 3 (Justified graph).** *A* justified graph *over an arena pair* $(A, B)$ *is a tuple* $s = (V_s, l_s, \underset{s}{\curvearrowright}, \underset{s}{\rightsquigarrow})$ *where:*

- $V_s$ *is a finite set called the* vertex set
- $l_s$ *is the* vertex labelling, *that is* $l_s \colon V_s \to \mathcal{M}_{A,B}$
- $\underset{s}{\curvearrowright} \subseteq V_s \times V_s$ *is the* justification relation
- $\underset{s}{\rightsquigarrow} \subseteq V_s \times V_s$ *is the* causality relation

*such that*

- $(V_s,\ \underset{s}{\curvearrowright} \cup \underset{s}{\rightsquigarrow})$ *is a DAG i.e. there is no cycle* $v\ (\underset{s}{\rightsquigarrow} \cup \underset{s}{\curvearrowright})^+\ v$.
- *If* $l_s(v)$ *is initial, then there is no node* $v'$ *such that* $v \underset{s}{\curvearrowright} v'$.
- *If* $l_s(v)$ *is not initial, then there exists a unique node* $v'$ *such that* $v \underset{s}{\curvearrowright} v'$. *Furthermore this* $v'$ *satisfies* $l_s(v') \vdash_{A,B} l_s(v)$.

*Note that* $\underset{s}{\curvearrowright}$ *and* $\underset{s}{\rightsquigarrow}$ *do not have to be disjoint. We define* $\underset{s}{\to} := (\underset{s}{\curvearrowright} \cup \underset{s}{\rightsquigarrow})$. *The set of justified graphs over an arena pair* $(A, B)$ *is denoted by* $J_{A,B}$.

In what follows, we shall identify isomorphic justified graphs.

Given a justified graph $s$ over $(A, B)$, a *P-node* (resp. an *O-node*) is a node $v \in V_s$ whose label is a P-move (resp. an O-move). We write $V_s^P$ for the set of P-nodes and $V_s^O$ for the set of O-nodes (e.g. $V_s^P := \{v \in V_s \mid \lambda_{A,B}(l_s(v)) = P\}$).

**Definition 4 (Play).** *Let* $s = (V_s, l_s, \underset{s}{\curvearrowright}, \underset{s}{\rightsquigarrow})$ *be a justified graph over* $(A, B)$. *It is a* play *if it satisfies the following conditions:*

**(P1)** *for every* $v, v' \in V_s$, $v \underset{s}{\rightsquigarrow} v'$ *implies* $v \in V_s^P$ *and* $v' \in V_s^O$,
**(P2)** *for every* $v_p \in V_s^P$ *and* $v_o \in V_s^O$, *if* $v_p \underset{s}{\to}^+ v_o$, *then* $v_p \underset{s}{\rightsquigarrow} v_o$, *and*
**(P3)** *for every* $v_o \in V_s^o$, *there exists* $v_p \in V_s^P$ *such that* $v_p \underset{s}{\rightsquigarrow} v_o$.

*We write* $P_{A,B}$ *for the set of plays over* $(A, B)$.

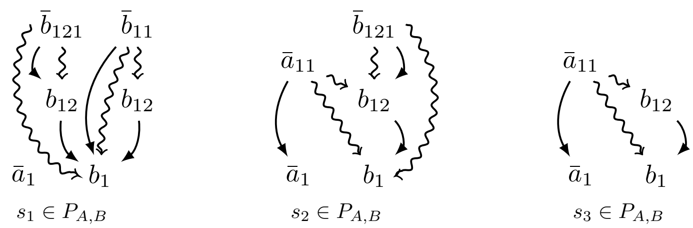



Fig. 4: Examples of plays over the arena pair $(A, B)$ in Figure 3.

Condition (P1) reflects the asynchronous nature of the target language. Recall that a P-move corresponds to an output action of a process and an O-move to an output action of the environment. No P-node should be causally related to P-nodes since an output action of the process cannot cause any other output of the process. Similarly no O-node should be causally related to O-nodes since an output action of the environment cannot cause any other output of the environment (provided that the environment is also described by the asynchronous $\pi$-calculus). An output action of a process may cause an output action of the environment; however it is a matter of the environment and a play describes the behaviour of a process, not the environment. Hence $\underset{s}{\rightsquigarrow} \subseteq V_s^P \times V_s^O$.

Condition (P2) comes from a purely technical requirement. (We need this condition to establish Lemma 2, as well as a proposition stating the copycat strategy is the identity.)

Condition (P3) is the counterpart of the even-length condition. Here we regard the even-length condition for sequential plays as the requirement that every O-move in the sequence should be responded by a P-move.

*Example 2.* Figure 4 shows three different plays over the arena pair $(A, B)$ in Figure 3. The solid arrows represent justification pointers, and squiggly arrows represent causalities. Nodes are labelled by moves and different nodes may be labelled by a same move. Note that plays may have a join point, i.e. a node that is linked to two "incomparable" nodes, like the node labelled by $\bar{a}_{11}$ in $s_2$.

*Remark 2.* A play can be seen as a process, e.g. the play $s_2$ in Figure 4 corresponds to the process $\nu(\bar{a}_{11}, a_{11}).(b_1(\_, b_{12}).b_{12}(\bar{b}_{121}, \_).(\bar{a}_{11} \mid \bar{b}_{121}) \mid \bar{a}_1\langle \_, a_{11}\rangle)$ (whose type differs from that described by the arena pair). The formal description of the connection to the linear internal $\pi$-calculus is left for the future work.

### 3.3 Strategies and composition

**Strategy** In most variants of sequential game models, a strategy $\sigma$ is a collection of plays that is *(even-length) prefix closed*: if $sm_O m_P \in \sigma$, then $s \in \sigma$. The set of strategies in our game model is defined by the same way, though the notion of prefix should be adapted to our setting.

**Definition 5 (Prefix).** *Let $s = (V_s, l_s, \underset{s}{\curvearrowright}, \underset{s}{\rightsquigarrow})$ be a play. Let $U \subseteq V_s$ be a subset that satisfies* (1) *$v \in U$ and $v \underset{s}{\rightarrow} v'$ implies $v' \in U$ and* (2) *for all $v_o \in U^O$ there* *exists* $v_p \in U^P$ *such that* $v_p \underset{s}{\rightsquigarrow} v_o$*. The* prefix $s[U] := (U, l, \curvearrowright, \rightsquigarrow)$ *of* $s$ *induced by* $U$ *is the restriction of* $s$ *to* $U$*, i.e.,*

$$l(v) := l_s(v) \quad \curvearrowright := (\underset{s}{\curvearrowright}) \cap (U \times U) \quad \rightsquigarrow := (\underset{s}{\rightsquigarrow}) \cap (U \times U).$$

*We write* $s' \sqsubseteq s$ *if* $s'$ *is a prefix of* $s$*. A prefix of a play is a play.*

*Example 3.* In Figure 4, the play $s_3$ is a prefix of $s_2$ induced by the set of nodes labelled by $m \in \{\bar{a}_1, \bar{a}_{11}, b_1, b_{12}\}$.

**Definition 6 (Strategy).** *Let* $(A, B)$ *be an arena pair. A set* $\sigma \subseteq P_{A,B}$ *of plays over* $(A, B)$ *is a* strategy *of* $(A, B)$*, written as* $\sigma \colon A \to B$*, if it satisfies* prefix-closedness *(S1):*

**(S1)** *If* $s \in \sigma$ *and* $s' \sqsubseteq s$*, then* $s' \in \sigma$*.*

**Composition** The composition of strategies is defined by using the notion of interactions. Since plays are not sequences but graphs, an interaction should also be represented by a graph that we call an *interaction graph*.

**Definition 7.** *Let* $(A, B, C)$ *be a triple of arenas. The set* $\mathcal{M}_{A,B,C}$ *of moves of* $(A, B, C)$ *is the disjoint union* $\mathcal{M}_A + \mathcal{M}_B + \mathcal{M}_C$*. The enabling relation* $\vdash_{A,B,C}$ *is defined by:* $x \vdash_{A,B,C} m$ *if* $x \vdash_X m$ *for some* $X \in \{A, B, C\}$*. The ownership function is defined by:* $\lambda_{A,B,C} := [\lambda_A, \lambda_B, \lambda_C]$*. The set* $J_{A,B,C}$ *of* justified graphs of $(A, B, C)$ *is defined by the same way as in Definition 3.*

For $X \in \{A, B, C, (A, B), (B, C), (A, C)\}$, we write $V_X$ for the set of nodes restricted to the component $X$ and $V_X^P$ and $V_X^O$ for the sets of nodes labelled by P-moves and by O-moves in the component $X$. For example, $v \in V_{A,B}^P$ means either (1) $l_u(v) \in \mathcal{M}_B$ and $\lambda_B(l_u(v)) = P$, or (2) $l_u(v) \in \mathcal{M}_A$ and $\lambda_A(l_u(v)) = O$.

**Definition 8 (Restriction).** *Let* $u = (V, l, \curvearrowright, \rightsquigarrow)$ *be a justified graph over* $(A, B, C)$ *and* $X \in \{(A, B), (B, C), (A, C)\}$*. The* restriction $u\restriction_X$ of $u$ to $X$ *is defined by* $u\restriction_X := (V\restriction_X,\ l\restriction_X,\ \curvearrowright\restriction_X,\ \rightsquigarrow\restriction_X)$*, where*

$$V\restriction_X := V_X, \quad l\restriction_X(v) := l(v), \quad \curvearrowright\restriction_X := (\curvearrowright) \cap (V_X \times V_X).$$

*The definition of* $\rightsquigarrow\restriction_X$ *needs some care. If* $X \in \{(A, B), (B, C)\}$*, then* $\rightsquigarrow\restriction_X$ *is just the restriction of the original causal relation, i.e.* $\rightsquigarrow\restriction_X := \{(v, v') \in V_X^P \times V_X^O \mid v \rightsquigarrow v'\}$ *(cf. Condition (P1)). If* $X = (A, C)$*, then* $\rightsquigarrow\restriction_{A,C}$ *relates moves linked through the intermediate component* $B$*, i.e.* $\rightsquigarrow\restriction_{A,C} := \{(v, v') \in V_{A,C}^P \times V_{A,C}^O \mid \exists n \geq 0.\ \exists v_1, \ldots, v_n \in V_B.\ v \rightsquigarrow v_1 \rightsquigarrow \cdots \rightsquigarrow v_n \rightsquigarrow v'\}$*.*

*Example 4.* Figure 5 shows a justified graph $u$ over the triple $(A, B, C)$ (in Figure 3) and its restrictions to components $(A, B)$, $(B, C)$ and $(A, C)$. Note that although $\bar{a}_{11} \underset{u}{\not\rightsquigarrow} c_1$, we have $\bar{a}_{11} \underset{u}{\rightsquigarrow}\restriction_{A,C} c_1$ because $\bar{a}_{11} \underset{u}{\rightsquigarrow} b_1 \underset{u}{\rightsquigarrow} c_1$.

**Definition 9 (Interaction graph).** *Let* $u \in J_{A,B,C}$ *be a justified graph over* $(A, B, C)$ *and* $V$ *be the set of nodes of* $u$*. We say that* $u$ *is an* interaction graph *if it satisfies the following conditions.*

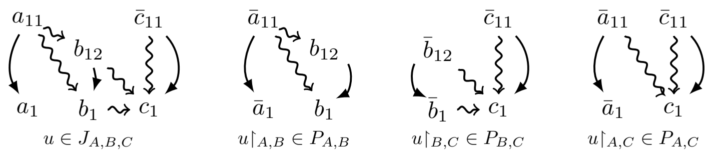



Fig. 5: Example of a justified graph and restrictions.

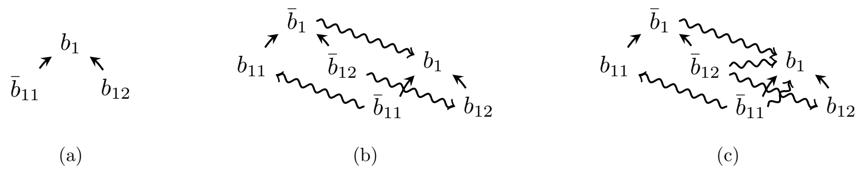



Fig. 6: Construction of a copycat play.

**(I1)** *If* $v \underset{u}{\rightsquigarrow} v'$, *then* $(v, v') \in V_X^P \times V_X^O$ *for some* $X \in \{(A, B), (B, C)\}$.
**(I2)** *Both* $u{\restriction}_{A,B}$ *and* $u{\restriction}_{B,C}$ *are plays.*

*Condition (I1) is a variant of the* switching condition. *The set of interaction graphs over* $(A, B, C)$ *is denoted as* $\mathrm{Int}(A, B, C)$.

In fact $u$ in Example 4 is an interaction graph.

**Definition 10 (Composition).** *Let* $\sigma\colon A \to B$ *and* $\tau\colon B \to C$ *be strategies. The* composition *of* $\sigma$ *and* $\tau$ *is defined by*

$$\tau \circ \sigma := \{u{\restriction}_{A,C} \mid u \in \mathrm{Int}(A, B, C),\ \ u{\restriction}_{A,B} \in \sigma,\ \ u{\restriction}_{B,C} \in \tau\}.$$

Note that the definition of composition is applicable to sets of plays that are not necessarily strategies. By abuse of notation, we shall write $\tau \circ \sigma$ even if $\sigma$ and $\tau$ are not strategies but just sets of plays.

**Theorem 1.** *The composite of strategies is a strategy. The composition is associative.*

**Category** We define the category $\mathcal{P}$ of negative arenas and strategies: an object of $\mathcal{P}$ is a (negative) arena and a morphism from $A$ to $B$ is a strategy $\sigma\colon A \to B$. The composite of $\sigma\colon A \to B$ and $\tau\colon B \to C$ is given by the composition $\tau \circ \sigma$ of strategies defined above. Given an arena $A$, the identity morphism $\mathrm{id}_A\colon A \to A$ is the "copycat strategy": when the environment makes a move $m$ in one component, then it responds by making a copy of $m$ in the other component. It is the set of *copycat plays*, whose construction is illustrated in Figure 6: $(a)$ take a "justified graph without causality" of the arena (in this example, the arena is $B$ in Figure 3); $(b)$ make positive and negative copies and connect the corresponding nodes by a causal edge $\rightsquigarrow$ in the appropriate direction; and $(c)$ add causal edges so as to satisfy Condition (P2), resulting in a play over $(B, B)$.

### 3.4 Distributive-closed Freyd category

In this section, we define the categorical structures of $\mathcal{P}$, which is used in Section 4 to give an interpretation of the $\pi$-calculus. A category with the structures below is called a *distributive-closed Freyd category* [24]. The definitions in this section are adapted from the interleaving game model for the $\pi$-calculus [24].

*Monoidal product* Let $A = (\mathcal{M}_A, \lambda_A, \vdash_A)$ and $B = (\mathcal{M}_B, \lambda_B, \vdash_B)$ be arenas. The arena $A \odot B$ is defined as $(\mathcal{M}_A + \mathcal{M}_B, [\lambda_A, \lambda_B], \vdash_{A,B})$, where $\vdash_{A,B}$ is the enabling relation defined in Definition 2. Given strategies $\sigma\colon A \to B$ and $\tau\colon C \to D$, the strategy $\sigma \odot \tau\colon A \odot C \to B \odot D$ is defined by the juxtaposition of plays in $\sigma$ and $\tau$, namely $\sigma \odot \tau := \{s \uplus t \mid s \in \sigma, t \in \tau\}$ where $s \uplus t$ is the juxtaposition of plays. Then the triple $(\mathcal{P}, \odot, I)$ is a symmetrical monoidal category.

*Closed Freyd structure* An input prefixing $a(\bar{\boldsymbol{x}}, \boldsymbol{y}).P$ should be interpreted by using a kind of closed structure (intuitively because the input prefix bounds variables in $P$ like $\lambda$-abstraction). Laird [24] used *closed Freyd categories* [33].

A Freyd category consists of a symmetric (pre)monoidal category $\mathcal{P}$, a cartesian category $\mathcal{A}$ and an identity-on-object strict (pre)monoidal functor $!\colon \mathcal{A} \to \mathcal{P}$. Intuitively $\mathcal{P}$ is that of types and "terms" whereas $\mathcal{A}$ is the category of types and "values"; the functor ! gives us a way to regard a "value" as a "term". In our context, "terms" are processes and "values" are processes of the form $\sum_i a_i(\bar{\boldsymbol{x}}_i, \boldsymbol{y}_i).P_i$, where $P_i$ has no free input channel except for those in $\boldsymbol{y}_i$.

We define the game-semantic counterpart of the processes of the this form.

**Definition 11 (Well-opened play, strategy).** *A play $s$ is* well-opened *if it contains precisely one initial O-node $v_0$ to which all other nodes are connected (i.e. $v \rightarrow_s^* v_0$ for every $v \in V_s$). We write $W_{A,B}$ for the set of well-opened plays over $(A, B)$. A* well-opened strategy *from arena $A$ to arena $B$, written as $\sigma\colon A \overset{\bullet}{\to} B$, is a set $\sigma$ of well-opened plays that is prefix-closed (S1).*

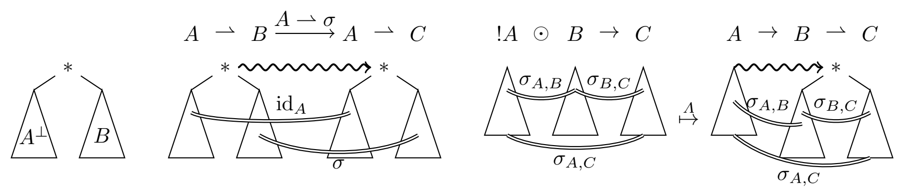
Then we define an operator !, a mapping from well-opened strategies to strategies and the composition of well-opened strategies by using !.

**Definition 12.** *Let $\sigma\colon A \overset{\bullet}{\to} B$ be a well-opened strategy. The strategy $!\sigma\colon A \to B$ is defined by $!\sigma := \{s_1 \uplus \cdots \uplus s_n \mid n \geq 0,\ \forall i \leq n.\, s_i \in \sigma\}$ where $s_1 \uplus \cdots \uplus s_n$ is the juxtaposition of plays $s_1, \ldots, s_n$.*

**Definition 13 (Composition of well-opened strategies).** *Let $\sigma\colon A \overset{\bullet}{\to} B$ and $\tau\colon B \overset{\bullet}{\to} C$ be well-opened strategies. We define $\tau \circ_{\mathcal{A}} \sigma := \tau \circ !\sigma$.*

**Lemma 1.** *The composite of well-opened strategies with respect to $\circ_{\mathcal{A}}$ is a well-opened strategy. The composition $\circ_{\mathcal{A}}$ of well-opened strategies is associative.*

The *category $\mathcal{A}$ of negative arenas and well-opened strategies* is defined by the following data: an object is a negative arena, a morphism from $A$ to $B$ is a well-opened strategy $\sigma\colon A \overset{\bullet}{\to} B$, the composition is given by $\circ_{\mathcal{A}}$. The identity morphism is $\mathrm{id}_A \cap W_{A,A}$, where $\mathrm{id}_A$ is the copycat strategy. The category $\mathcal{A}$ is cartesian: the cartesian product of $A$ and $B$ is $A \odot B$.




(a) $A \rightharpoonup B$. (b) $A \rightharpoonup \sigma$. (c) $\Lambda(\sigma)$.

Fig. 7: The action of $A \rightharpoonup (-)$ and $\Lambda$.

By defining $!A := A$ for objects, the operation $!$ becomes a functor $!\colon \mathcal{A} \to \mathcal{P}$. This is identity on objects and strict symmetric monoidal functor and thus $(\mathcal{A}, \mathcal{P}, !)$ is a Freyd category.

**Lemma 2.** *The Freyd category $(\mathcal{A}, \mathcal{P}, !)$ is closed, i.e. for every arena $A$, the functor $!(-) \odot A\colon \mathcal{A} \to \mathcal{P}$ has the right-adjoint $A \rightharpoonup (-)\colon \mathcal{P} \to \mathcal{A}$.*

The action of $A \rightharpoonup (-)$ on objects and on morphisms is illustrated in Figure 7. We write $\Lambda$ for the bijective map $\mathcal{P}(!A \odot B, C) \to \mathcal{A}(A, B \rightharpoonup C)$ and $\mathbf{app}_{A,B}\colon !(A \rightharpoonup B) \odot A \to B$ for the counit. The bijection $\mathcal{P}(!A \odot B, C) \cong \mathcal{A}(A, B \rightharpoonup C)$ induced by the adjunction intuitively corresponds to the following bijection of the $\pi$-calculus processes: $\bar{\boldsymbol{x}} : \boldsymbol{S}, \bar{\boldsymbol{y}} : \boldsymbol{T} \vdash P; \boldsymbol{z} : \boldsymbol{U} \quad \longleftrightarrow \quad \bar{\boldsymbol{x}} : \boldsymbol{S} \vdash a(\bar{\boldsymbol{y}}, \boldsymbol{z}).P; a : \mathbf{ch}[\boldsymbol{T}, \boldsymbol{U}]$.

*Distributive law* The process obtained by (the $\pi$-term representation of) the above adjunction has the input prefix $a(\bar{\boldsymbol{y}}, \boldsymbol{z})$ as expected but it has only one free input channel. We use the *distributive law* of the distributive-closed Freyd category to model a process with multiple free input channel. By using the syntax of the $\pi$-calculus, the distribution law can be seen as the following map:

$$\bar{\boldsymbol{x}}\colon \boldsymbol{S} \vdash a(\bar{\boldsymbol{y}}, \boldsymbol{z}\boldsymbol{z}').P;\ a\colon \mathbf{ch}[\boldsymbol{T}, \boldsymbol{U}\boldsymbol{U}'] \ \longrightarrow\ \bar{\boldsymbol{x}}\colon \boldsymbol{S} \vdash a(\bar{\boldsymbol{y}}, \boldsymbol{z}).P;\ a\colon \mathbf{ch}[\boldsymbol{T}, \boldsymbol{U}], \boldsymbol{z}'\colon \boldsymbol{U}'.$$

**Definition 14 (Distributive-closed Freyd category [24]).** *A closed Freyd category $!\colon \mathcal{A} \to \mathcal{P}$ is* distributive-closed *if there is a family of morphisms $\varrho_A\colon !(A \rightharpoonup (B \odot C)) \longrightarrow B \odot !(A \rightharpoonup C)$ in $\mathcal{P}$, natural in $B$ and $C$ which makes certain diagrams commute.*

**Theorem 2.** *The game model $!\colon \mathcal{A} \to \mathcal{P}$ is distributive-closed.*

*Trace* The operator $\nu(\bar{x}, y).P$ is interpreted as a trace operator. We define $Tr^B_{A,C}(f) := \mathbf{app}_{B,C} \circ \mathbf{symm}_{B,B \rightharpoonup C} \circ \varrho_{B,B,C} \circ !\Lambda(\mathbf{symm}_{B,C} \circ f)$, given a morphism $f\colon A \odot B \to C \odot B$ in $\mathcal{P}$. Then $Tr$ is the trace operator for the symmetrical monoidal category $\mathcal{P}$ [24].

*Additional structures* Some additional structures are required to interpret the $\pi$-calculus: the *minimum strategy* $\perp_{A,B}$ (with respect to the set-inclusion), the *diagonal* $\Delta_A\colon A \overset{\bullet}{\to} A \odot A$, the *codiagonal* $\nabla_A\colon A \odot A \overset{\bullet}{\to} A$ (defined by $\nabla_A := \pi_1 \cup \pi_2$ where $\pi_i\colon A \odot A \overset{\bullet}{\to} A$ is the projection), and the *dereliction* $der_A\colon A \to A$ (defined as $\mathrm{id}_A \cap W_{A,A}$).

### 3.5 Relation to sequential game models

*Laird's interleaving game model* Our model can be seen as a truly concurrent version of the interleaving game model $\mathcal{P}_{\mathrm{L}}$ of Laird [24]. The idea is to relate a (concurrent) play $s = (V_s, l_s, \underset{s}{\curvearrowleft}, \underset{s}{\rightsquigarrow})$ to an interleaving play by lining up the nodes in $V_s$ in such a way that if $v_1 \underset{s}{\rightsquigarrow} v_2$, then $v_2$ appears before $v_1$. We write $|s|$ for the set of sequential plays obtained by this way.

*Example 5.* Let $s_2$ be the play in Figure 4. Then $|s_2|$ is given as:

$$\left\{ \begin{array}{c} \bar{a}_1\, b_1\, b_{12}\, \bar{a}_{12}\, \bar{b}_{121}\,, \quad \bar{a}_1\, b_1\, b_{12}\, \bar{b}_{121}\, \bar{a}_{12}\,, \quad b_1\, \bar{a}_1\, b_{12}\, \bar{a}_{12}\, \bar{b}_{121}\,, \quad b_1\, \bar{a}_1\, b_{12}\, \bar{b}_{121}\, \bar{a}_{12}\,, \\ b_1\, b_{12}\, \bar{a}_1\, \bar{a}_{12}\, \bar{b}_{121}\,, \qquad b_1\, b_{12}\, \bar{a}_1\, \bar{b}_{121}\, \bar{a}_{12}\,, \qquad b_1\, b_{12}\, \bar{b}_{121}\, \bar{a}_1\, \bar{a}_{12} \end{array} \right\}$$

**Theorem 3.** $|-|$ *induces an identity-on-object functor from $\mathcal{P}$ to $\mathcal{P}_{\mathrm{L}}$, which preserves the structure of distributed-closed Freyd categories (and the additional structures). Furthermore $|\sigma|$ is the minimum strategy if and only if so is $\sigma$.*

*Sequential HO/N game model* The standard sequential HO/N game model [21] is a subcategory of our concurrent model. Since our game model only have question moves, we compare our model with the HO/N game model without *answer* (and thus without *well-bracketing*).

An arena $A$ is *alternating* if $m \vdash_A m'$ implies $\lambda_A(m) = \lambda_A^{\perp}(m')$. Let $\mathcal{G}$ be the category of negative alternating arenas and innocent strategies (we omit the definition, which is standard). We write $\ulcorner \hat{s} \urcorner$ for the *P-view* [21] of the sequential play $\hat{s}$. Given a sequential play $\hat{s} = m_1 \ldots m_n$, a DAG-based play is given by

$$\|\hat{s}\| := (V_{\hat{s}}, l_{\hat{s}},\;\; \{(i,j) \mid \rho_{\hat{s}}(i) = j\},\;\; \{(i,j) \in V_{\hat{s}}^P \times V_{\hat{s}}^O \mid m_j \in \ulcorner m_1 \ldots m_i \urcorner\})$$

where $V_{\hat{s}} := \{1, \ldots, n\}$, $l_{\hat{s}}(i) := m_i$ and $\rho_{\hat{s}}$ is the partial function describing the justification pointer. Note that the occurrence $m_i$ of a P-move is causally related to an occurrence $m_j$ of an O-move if and only if $m_j$ appears in the P-view of $m_i$. This map is naturally extended to strategies, namely $\|\hat{\sigma}\| := \{\|\hat{s}\| \mid \hat{s} \in \hat{\sigma}\}$.

**Theorem 4.** $\|-\|$ *induces a faithful functor from $\mathcal{G}$ to $\mathcal{P}$.*

*Remark 3.* It is natural to ask if one can give a similar map from Laird's interleaving model. The answer seems negative: all maps that we have checked are not functorial. See [8] for a related result.

## 4 Game Semantics of the $\pi$-calculus

We give an interpretation of the $\pi$-calculus, following the result of Laird [24] applicable to every distributive-closed Freyd category with additional structures.

A type and a type environment are interpreted as objects of $\mathcal{P}$. The interpretation of a type $\mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}]$ and a sequence $\boldsymbol{S}$ of types are defined by:

$$[\![\mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}]]\!] := [\![\boldsymbol{S}]\!] \rightharpoonup [\![\boldsymbol{T}]\!] \qquad [\![S_1 \ldots S_n]\!] := [\![S_1]\!] \odot \cdots \odot [\![S_n]\!] \qquad [\![\_]\!] := I.$$

$$\llbracket \Gamma \vdash \mathbf{0}; \Sigma \rrbracket = \bot_{\llbracket \Gamma \rrbracket, \llbracket \Sigma \rrbracket}$$
$$\llbracket \Gamma, \Gamma' \vdash P | Q; \Sigma, \Sigma' \rrbracket = \llbracket P \rrbracket \odot \llbracket Q \rrbracket$$
$$\llbracket \Gamma, \bar{x} : \mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}], \bar{\boldsymbol{y}} : \boldsymbol{S} \vdash \bar{x}\langle \bar{\boldsymbol{y}}, \boldsymbol{z} \rangle; \Sigma, \boldsymbol{z} : \boldsymbol{T} \rrbracket = \bot_{\llbracket \Gamma \rrbracket, \llbracket \Sigma \rrbracket} \odot \mathbf{app}_{\llbracket \boldsymbol{S} \rrbracket, \llbracket \boldsymbol{T} \rrbracket}$$
$$\llbracket \Gamma \vdash x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P; \Sigma, x : \mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}] \rrbracket = (\mathrm{id}_{\llbracket \Sigma \rrbracket} \odot der_{\llbracket (\boldsymbol{S}, \boldsymbol{T}) \rrbracket}) \circ \llbracket !x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P \rrbracket$$
$$\llbracket \Gamma \vdash !x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P; \Sigma, x : \mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}] \rrbracket = \varrho_{\llbracket \boldsymbol{S} \rrbracket, \llbracket \Sigma \rrbracket, \llbracket \boldsymbol{T} \rrbracket} \circ !\Lambda(\llbracket P \rrbracket)$$
$$\llbracket \Gamma \vdash \nu(\bar{x}, y).P; \Sigma \rrbracket = Tr^{\llbracket T \rrbracket}_{\llbracket \Gamma \rrbracket, \llbracket \Sigma \rrbracket}(\llbracket P \rrbracket)$$

Fig. 8: Interpretation of processes. (Contraction and exchange rules are omitted.)

The interpretation of an input type environment is given by the tensor product of elements, e.g. $\llbracket x_1 : S_1, \ldots, x_n : S_n \rrbracket := \llbracket S_1 \rrbracket \odot \cdots \odot \llbracket S_n \rrbracket$.

A process $\Gamma \vdash P; \Sigma$ is interpreted as a morphism $\llbracket P \rrbracket \colon \llbracket \Gamma \rrbracket \to \llbracket \Sigma \rrbracket$ in $\mathcal{P}$. The interpretation is defined by induction on the type derivations. The rules are listed in Figure 8.

The distributive-closed Freyd structure together with additional structures (of $\Delta$, $\nabla$, $\bot$, $der$) gives a (weak) soundness result with respect to the reduction.

**Theorem 5.** *Let $\Gamma \vdash P; \Sigma$ and $\Gamma \vdash Q; \Sigma$ be processes of the same type.*

1. *If $P \equiv Q$, then $\llbracket \Gamma \vdash P; \Sigma \rrbracket = \llbracket \Gamma \vdash Q; \Sigma \rrbracket$.*
2. *If $P \longrightarrow Q$, then $\llbracket \Gamma \vdash P; \Sigma \rrbracket \supseteq \llbracket \Gamma \vdash Q; \Sigma \rrbracket$.*

The relationship to Laird's model (Theorem 3) gives a finer result, which does not follow from the general theory of the distributive-closed Freyd categories.

**Lemma 3 (Adequacy).** *$P \Downarrow_{\bar{x}}$ iff $\llbracket P \rrbracket \neq \bot$ for every process $\bar{x} : \mathbf{ch}[] \vdash P; \_$.*

*Proof.* Because of Theorem 3, we have $|\llbracket P \rrbracket| = \llbracket P \rrbracket_L$, where $\llbracket P \rrbracket_L$ is the interpretation of the process in Laird's game model [24]. Laird [24] shows that $P \Downarrow_{\bar{x}}$ if and only if $\llbracket P \rrbracket_L \neq \bot$. Since $|-|$ preserves $\bot$, we obtain the claim. □

Lemma 3 and monotonicity of the interpretation lead to the next theorem.

**Theorem 6.** *Let $\bar{x}$ be a testing name that does not occur in $\Gamma$. If $\llbracket \Gamma \vdash P; \Sigma \rrbracket \subseteq \llbracket \Gamma \vdash Q; \Sigma \rrbracket$, then $C[P] \Downarrow_{\bar{x}}$ implies $C[Q] \Downarrow_{\bar{x}}$ for all context $C[\,]$.*

Unlike Laird's model [24], our model is not complete since our model is truly concurrent. For example, $\llbracket a().\bar{b}\langle\rangle \mid c().\bar{d}\langle\rangle \rrbracket \neq \llbracket \nu(\bar{x}, x).(\bar{x}\langle\rangle \mid x().a().(\bar{b}\langle\rangle \mid c().\bar{d}\langle\rangle) \mid x().c().(a().\bar{b}\langle\rangle \mid \bar{d}\langle\rangle)) \rrbracket$ in our model, whereas they are testing equivalent.

## 5 Discussion: Relationally-Describable Process

Using our game model, we study the relational interpretations of process in the form of intersection type system that describes the behaviour of processes. The

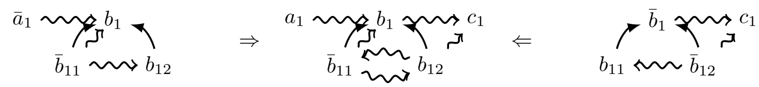



Fig. 9: Composable plays with a cycle.

intersection type system is a fully abstract model for *a class of* process which we characterise with the help of "interaction graph".

The syntax of *types* and *intersections* are defined by the following grammar:

$$\varphi, \psi ::= \mathbf{ch}[\xi_1 \dots \xi_n, \zeta_1 \dots \zeta_k] \qquad \xi, \zeta ::= \langle \varphi_1, \dots, \varphi_n \rangle$$

where $\langle \cdots \rangle$ is a finite multiset defined by an enumeration of the elements. A *type environment* is a sequence of type bindings of the form $x : \xi$ (or $\bar{y} : \zeta$). Given intersections $\xi = \langle \varphi_1, \dots, \varphi_n \rangle$ and $\zeta = \langle \psi_1, \dots, \psi_k \rangle$, we write $\xi \wedge \zeta$ for $\langle \varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_k \rangle$. This operation is extended to type environments by pointwise application. The typing rules are listed below (some rules are omitted):

$$\frac{}{\emptyset \vdash \mathbf{0};\ \emptyset} \quad \frac{\Xi \vdash P;\ \Theta \qquad \Xi' \vdash P';\ \Theta'}{\Xi, \Xi' \vdash P|P';\ \Theta, \Theta'} \quad \frac{}{\bar{x} : \mathbf{ch}[\boldsymbol{\xi}, \boldsymbol{\zeta}], \bar{\boldsymbol{y}} : \boldsymbol{\xi} \vdash \bar{x}\langle \bar{\boldsymbol{y}}, \boldsymbol{z} \rangle;\ \Theta, \boldsymbol{z} : \boldsymbol{\zeta}}$$

$$\frac{\Xi, \bar{\boldsymbol{y}}{:}\boldsymbol{\xi} \vdash x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P;\ \Theta, \boldsymbol{z}{:}\boldsymbol{\zeta}}{\Xi \vdash x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P;\ \Theta, x{:}\mathbf{ch}[\boldsymbol{\xi}, \boldsymbol{\zeta}]} \quad \frac{\forall i \in I.\ \Xi_i \vdash x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P;\ \Theta_i}{\bigwedge_{i \in I} \Xi_i \vdash !x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P;\ \bigwedge_{i \in I} \Theta_i} \quad \frac{\Xi, \bar{x}{:}\xi \vdash P;\ \Theta, y{:}\xi}{\Xi \vdash \nu(\bar{x}, y).P;\ \Theta}$$

This type system is inspired by the correspondence between intersection type systems and the operation called *time forgetting map* [4], which is an operation that forgets the temporal structure of plays, in sequential game models (see, e.g., [37]). Time forgetting map is the operation that forgets the causal relation in the case of our concurrent game model.

Completeness of the type system holds for every process, but soundness does not; the reason is explained by a game-semantic consideration. We would thus like to find a class for which the relational interpretation is sound.

Let $s \in P_{A,B}$ and $t \in P_{B,C}$ be plays. We say that $s$ and $t$ are *composable* if $s{\restriction_B}$ coincides with $t{\restriction_B}$ except for the causal relations. Then it would be natural to think of an "interaction graph" by composing them (see Fig. 9). Unfortunately the resulting "interaction graph" may not be acyclic and hence not be an interaction graph; in this case we say that the pair $(s, t)$ contains a cycle.

This notion of cycle can be extended to strategies and processes. The composition of strategies $\tau \circ \sigma$ is *cycle-free* if every pair of composable plays $s \in \sigma$ and $t \in \tau$ is cycle-free. A process $P$ is *relationally-describable* if every composition in the definition of $[\![P]\!]$ is cycle-free.

**Theorem 7.** *Let $\Gamma \vdash P;\ \Sigma$ be a relationally-describable process and let $\bar{x} \in \mathrm{dom}(\Gamma)$. Then $P \Downarrow_{\bar{x}}$ if and only if $\bar{x} : \mathbf{ch}[\boldsymbol{\xi}, \boldsymbol{\zeta}] \vdash P;\ \emptyset$ for some $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$.*

This is because the operation of forgetting the causal relation commutes with cycle-free composition. Note that the notion of cycle is stronger than deadlock: $\nu(\bar{a}\bar{b}, ab).(a_1.\bar{b}_2|b_3.\bar{a}_4|\bar{a}_5)$ (subscripts are used to distinguish occurrences) is *not* relationally-describable because connecting $a_1$ to $\bar{a}_4$ and $b_3$ to $\bar{b}_2$ creates a cycle.

Restricting the form of processes by focusing on cycles is a reminiscent of the correctness criterion for MLL proof nets. The formal relationship between our notion of cycle in an interaction graph and the correctness criterion, and the connection between cycle (in our sense) and the type system, which gives a typed $\pi$-calculus corresponding to polarised proof-nets satisfying the correctness criterion, proposed by Honda and Laurent [19] are worth investigating.

## 6 Related Work

Melliès [27] studied HO/N innocent strategies from a truly concurrent point of view. Among others, he introduced the notions of *alternating homotopy* and *diagrammatic innocence*, which influence to this work. These ideas were subsequently developed by Melliès and Mimram [29,30], who introduced *asynchronous games*. They focused on the fact that some moves of a play in an innocent strategy can be exchanged, and studied games whose rules explicitly describe which moves should be commutable. Our game model is also inspired by [27] (and [28]) but we focused on a different aspect of the alternating homotopy, that is, the fact that the connection between a successive pair of O- and P-moves (in HO/N innocent strategies) are quite tight (see also [25,36]); in our game model, a strategy explicitly describes indispensable connections $\rightsquigarrow$ between events. Because of these differences, their game model differs from ours; indeed our strategy is not necessarily *positional*. Nevertheless those models seems closely related; for example, it seems worth investigating the connection between *scheduled strategies* [30] and cycle-free composition.

A related approach using a map of *event structures* has been proposed by Rideau and Winskel [34] and extensively studied recently [9, 10]. In this game model, a strategy is a map from an event structure describing the internal causal relation to another event structure expressing the observable events. We think that their model should be closely related to the (pre)sheaf version [36] of our game model, although we have not established any formal relationship yet.

From a technical point of view, an important difference between above models and our model is the way to deal with duplication of moves. Our model uses HO/N-style justification pointers, whereas the above models use the idea of *thread indexing* [10, 26] in the style of AJM game model [1]. Both approaches have advantages and disadvantages (for example, an advantage of the HO/N-style is that a morphism is a strategy, not an equivalence class of strategies modulo reindexing). Hence we think that it is good to have a truly concurrent model using justification pointers.

Laird [24] briefly discussed an idea of a truly concurrent version of his interleaving game model, introducing the notion of *justified pomset*. His idea is very closed to ours; indeed a play $s$ in our game model can be seen as a pomset $(V_s, \underset{s}{\rightarrow}^*)$ ordered by (reflexive transitive closure of) the adjacent relation $\underset{s}{\rightarrow}^*$.

A DAG-based reformulation of the HO/N game model is a reminiscent of *$L$-nets* [13,17]. The conditions required for $L$-nets are essentially the same as those we require for plays, though $L$-nets corresponds to strategies, not to plays. An interpretation of the $\pi$-calculus using *differential nets* [15] seems to be relevant to our development.

The game-semantics study of this paper has many parallels to the syntactic study of the $\pi$-calculus. The relationship between the HO/N game model for PCF [21] and the $\pi$-calculus has originally been studied by Hyland and Ong themselves [20], who gave a translation from PCF terms to processes of the $\pi$-calculus based on the idea of their game model. The $\pi$-terms representing sequential functional computation can be characterised by a simple type system proposed by Berger, Honda and Yoshida [5], which lead to the type system of [19]. We conjecture that processes typed by the simple type system of [5,19] is related to relationally-describable processes. Boreale [6] gave an encoding from the asynchronous $\pi$-calculus to the *internal $\pi$-calculus* [35]. Our game model can be seen as a variant of the encoding by regarding the plays as the processes of the *linear* internal $\pi$-calculus, in which each name must be used exactly once.

There are some pieces of work based on the techniques other than games but related to this work, such as event structure semantics of several variants of the $\pi$-calculus by Crafa, Varacca and Yoshida [11,12] and Varacca and Yoshida [38], and a data-flow semantics by Jagadeesan and Jagadeesan [22].

## 7 Conclusion and Future Work

We have developed a truly concurrent version of the HO/N game model [21,32], in which a computation is represented by a DAG of messages instead of a sequence. The resulting game model has the categorical structure needed to interpret the asynchronous $\pi$-calculus proposed by Laird [24]. By using the connection between our model and Laird's model [24], we have proved soundness of the interpretation of the processes in our concurrent game model. This is the first truly concurrent game semantics for the $\pi$-calculus.

We have several topics left for future work:

- Formal description of the connection between plays and processes mentioned in Remark 2. By this connection, our game semantics can be seen as an approximation of the processes of the $\pi$-calculus by a linear $\pi$-calculus, which is a reminiscent of the *Taylor expansion* of the $\lambda$-calculus [16] (see also [37]).
- Development of the (pre)sheaf version of the game model [36], which would be related to the game model based on [34].
- Development of a model of the synchronous $\pi$-calculus. This requires us to deal with causal edges from O-moves and/or to P-moves. To simply relax the requirements for $\rightsquigarrow$ does not seem to work: for example, the copycat strategy of this paper is no longer the identity in the relaxed version.
- Development of a model of the $\pi$-calculus with the matching primitive. We expect that a nominal game model [31] would be useful for this purpose.

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$$\frac{\Gamma, \bar{x} : T, \bar{y} : T \vdash P; \Sigma}{\Gamma, \bar{z} : T \vdash P\{\bar{z}/\bar{x}, \bar{z}/\bar{y}\}; \Sigma} \qquad \frac{\Gamma \vdash P; \Sigma, x : T, y : T}{\Gamma \vdash P\{z/x, z/y\}; \Sigma, z : T}$$

$$\frac{\Gamma, \bar{x} : S, \bar{y} : T, \Gamma' \vdash P; \Sigma}{\Gamma, \bar{y} : T, \bar{x} : S, \Gamma' \vdash P; \Sigma} \qquad \frac{\Gamma \vdash P; \Sigma, x : S, y : T, \Sigma'}{\Gamma \vdash P; \Sigma, y : T, x : S, \Sigma'}$$

$$\frac{}{\Gamma \vdash \mathbf{0}; \Sigma} \qquad \frac{\Gamma \vdash P; \Sigma \qquad \Gamma' \vdash Q; \Sigma'}{\Gamma, \Gamma' \vdash P|Q; \Sigma, \Sigma'}$$

$$\frac{}{\Gamma, \bar{x} : \mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}], \bar{\boldsymbol{y}} : \boldsymbol{S} \vdash \bar{x}\langle \bar{\boldsymbol{y}}, \boldsymbol{z}\rangle; \Sigma, \boldsymbol{z} : \boldsymbol{T}} \quad \frac{\Gamma, \bar{\boldsymbol{y}} : \boldsymbol{S} \vdash P; \Sigma, \boldsymbol{z} : \boldsymbol{T}}{\Gamma \vdash x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P; \Sigma, x : \mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}]}$$

$$\frac{\Gamma, \bar{\boldsymbol{y}} : \boldsymbol{S} \vdash P; \Sigma, \boldsymbol{z} : \boldsymbol{T}}{\Gamma \vdash !x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P; \Sigma, x : \mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}]} \qquad \frac{\Gamma, \bar{x} : T \vdash P; \Sigma, y : T}{\Gamma \vdash \nu(\bar{x}, y).P; \Sigma}$$

Fig. 10: The complete list of typing rules.

# A Supplementary Material for Section 2

**Structural congruence** The *structural congruence* is the least congruence relation that subsumes the $\alpha$-equivalence and the following rules:

$$P|Q \equiv Q|P \qquad \mathbf{0}|P \equiv P \qquad (P|Q)|R \equiv P|(Q|R) \qquad !P \equiv P|!P$$
$$\nu(\bar{x}_1, y_1).\nu(\bar{x}_2, y_2).P \equiv \nu(\bar{x}_2, y_2).\nu(\bar{x}_1, y_1).P \qquad (\nu(\bar{z}, w).P)|Q \equiv \nu(\bar{z}, w).(P|Q)$$

where $\bar{x}_1 \neq \bar{x}_2$, $y_1 \neq y_2$ and $\bar{z}, w \notin \mathbf{fn}(Q)$.

**Reduction relation** The *one-step reduction relation* $\longrightarrow$ on processes is defined by the following rule:

$$\nu(\bar{\boldsymbol{z}}, \boldsymbol{w}).\nu(\bar{x}, y).(y(\bar{\boldsymbol{a}}, \boldsymbol{b}).P \mid \bar{x}\langle \bar{\boldsymbol{c}}, \boldsymbol{d}\rangle \mid Q) \longrightarrow \nu(\bar{\boldsymbol{z}}, \boldsymbol{w}).\nu(\bar{x}, y).(P\{\bar{\boldsymbol{c}}/\bar{\boldsymbol{a}}, \boldsymbol{d}/\boldsymbol{b}\} \mid Q)$$

It should be noted that the communication only occurs over names that are bound by $\nu$. We define the *reduction relation* over processes as the reflexive transitive closure of $(\longrightarrow \cup \equiv)$.

**Complete list of typing rules** Because of the space limitation, we have omitted some typing rules in the body of the paper. Figure 10 is the complete list of typing rules.

# B Supplementary Materials on the Category $\mathcal{P}$ (Sections 3.1, 3.2 and 3.3)

## B.1 Notations

We denote $\mathcal{M}_A^O$ for the set of *O-moves* $\{m \in \mathcal{M}_A \mid \lambda_A(m) = O\}$ and $\mathcal{M}_A^P$ for the set of *P-moves* $\{m \in \mathcal{M}_A \mid \lambda_A(m) = P\}$. We write $I_A$ for the set of initial moves in an arena $A$. Give an arena $A = (\mathcal{M}_A, \lambda_A, \vdash_A)$ the negation of $A$ is defined by $A^\perp := (\mathcal{M}_A, \lambda_A^\perp, \vdash_A)$.

### B.2 Isomorphism of justified graphs

In the body of the paper, we wrote that justified graphs are identified up to isomorphism, but did not give the definition of the isomorphism between justified graphs. The following is the definition.

**Definition 15.** *Let $s = (V_s, l_s, \underset{s}{\curvearrowright}, \underset{s}{\rightsquigarrow})$ and $t = (V_t, l_t, \underset{t}{\curvearrowright}, \underset{t}{\rightsquigarrow})$ be justified graphs over $(A, B)$. We say that $s$ and $t$ are* isomorphic *if there exists a bijection $\varphi : V_s \to V_t$ that preserves and reflects all the structures (i.e. $l_s(v) = l_t(\varphi(v))$, $v \underset{s}{\curvearrowright} v' \Leftrightarrow \varphi(v) \underset{t}{\curvearrowright} \varphi(v')$ and $v \underset{s}{\rightsquigarrow} v' \Leftrightarrow \varphi(v) \underset{t}{\rightsquigarrow} \varphi(v')$).*

### B.3 Justified graphs over tuples of arenas

In the body of the paper, we have introduced justified graphs and plays over a pair of arenas, as well as justified and interaction graphs over a triple of arenas. We generalise these notions to a tuple of arenas (of length greater than 1).

**Definition 16 (Tuple of arenas).** *Let $\boldsymbol{A} = (A^1, \dots, A^n)$ be an tuple of arenas. The set $\mathcal{M}_{\boldsymbol{A}}$ of* moves of $\boldsymbol{A}$ *is defined by $\mathcal{M}_{\boldsymbol{A}} = \mathcal{M}_{A^1} + \dots + \mathcal{M}_{A^n}$. The enabling relation $\vdash_{\boldsymbol{A}}$ is defined by: $m \vdash_{\boldsymbol{A}} m'$ if $m \vdash_A m'$ for some $A \in \{A^1, \dots, A^n\}$. The ownership function is defined by $\lambda_{\boldsymbol{A}} = [\lambda_{A^1}, \dots, \lambda_{A^n}]$.*

The definition of justified graphs is the straightforward extension of that over a pair (or a triple) of arenas.

**Definition 17 (Justified graph).** *A* justified graph of $\boldsymbol{A}$ *is a tuple $s = (V_s, l_s, \underset{s}{\curvearrowright}, \underset{s}{\rightsquigarrow})$ where*

- *$V_s$ is a finite set called the* vertex set
- *$l_s$ is the* vertex labelling, *that is $l_s \colon V_s \to \mathcal{M}_{\boldsymbol{A}}$*
- *$\underset{s}{\curvearrowright} \subseteq V_s \times V_s$ is a* justification relation
- *$\underset{s}{\rightsquigarrow} \subseteq V_s \times V_s$ is a* causality relation

*such that*

- *$(V_s, \underset{s}{\curvearrowright} \cup \underset{s}{\rightsquigarrow})$ is a DAG i.e. there is no cycle $v \ (\underset{s}{\rightsquigarrow} \cup \underset{s}{\curvearrowright})^+ \ v$.*
- *If $l_s(v)$ is initial, then there is no node $v'$ such that $v \underset{s}{\curvearrowright} v'$.*
- *If $l_s(v)$ is not initial, then there exists a unique node $v'$ such that $v \underset{s}{\curvearrowright} v'$. Furthermore this $v'$ satisfies $l_s(v') \vdash_{\boldsymbol{A}} l_s(v)$.*

*We write $J_{\boldsymbol{A}}$ for the set of justified graphs over $\boldsymbol{A}$.*

We define $\underset{s}{\rightarrow} := (\underset{s}{\curvearrowright} \cup \underset{s}{\rightsquigarrow})$.

Given a tuple $\boldsymbol{A} = (A^1, \dots, A^n)$ of arenas, its *subsequence* is of the form $(A^{i_1}, A^{i_2}, \dots, A^{i_k})$ where $0 \leq i_1 < i_2 < \dots < i_k \leq n$ is a strictly increasing sequence of indexes. We write $\boldsymbol{A} \setminus \boldsymbol{B}$ for the subsequence of $\boldsymbol{A}$ consisting of arenas not in $\boldsymbol{B}$ (e.g. $(A, B, C, D) \setminus (B, D) = (A, C)$).

Let $\boldsymbol{A}$ be a tuple of arenas, $\boldsymbol{B}$ be a subsequence of $\boldsymbol{A}$ and $u = (V, l, \curvearrowright, \rightsquigarrow)$ be a justified graph over $\boldsymbol{A}$. Then we define:

$$
\begin{aligned}
V_{\boldsymbol{B}} &:= \{v \in V \mid l(v) \in \mathcal{M}_{\boldsymbol{B}}\} \\
V^{O}_{A^i} &:= \{v \in V \mid l(v) \in \mathcal{M}^{O}_{A^i}\} \\
V^{P}_{A^i} &:= \{v \in V \mid l(v) \in \mathcal{M}^{P}_{A^i}\} \\
V^{O}_{A^i,A^j} &:= \{v \in V \mid l(v) \in \mathcal{M}^{P}_{A^i} + \mathcal{M}^{O}_{A^j}\} \\
V^{P}_{A^i,A^j} &:= \{v \in V \mid l(v) \in \mathcal{M}^{O}_{A^i} + \mathcal{M}^{P}_{A^j}\},
\end{aligned}
$$

where $i < j$. For $v, v' \in V$ and $W \subseteq V$ with $v \neq v'$, we write $v \rightsquigarrow W^* \rightsquigarrow v'$ if there exists a sequence

$$v = v_0 \rightsquigarrow v_1 \rightsquigarrow \ldots \rightsquigarrow v_{\ell-1} \rightsquigarrow v_\ell = v'$$

such that $v_i \in W$ for every $i \in \{1, 2, \ldots, \ell - 1\}$. Note that $v \rightsquigarrow W^* \rightsquigarrow v'$ implies $v \rightsquigarrow^+ v'$.

We also generalise the notion of prefix.

**Definition 18 (Induced subgraph).** *Given a justified graph* $s = (V_s, l_s, \underset{s}{\curvearrowright}, \underset{s}{\rightsquigarrow})$ *and* $W \subseteq V_s$*, we write* $s[W] = (W, l, \curvearrowright, \rightsquigarrow)$ *for the labelled subgraph of* $s$ *induced by* $W$ *i.e.*

$$l(v) := l_s(v) \qquad \curvearrowright := (\underset{s}{\curvearrowright}) \cap (W \times W) \qquad \rightsquigarrow := (\underset{s}{\rightsquigarrow}) \cap (W \times W)$$

*The subgraph* $s[W]$ *of* $s$ *is also a justified graph.*

**Definition 19 (Restriction).** *Let* $\boldsymbol{A} = (A^1, \ldots, A^n)$ *be a tuple of arenas and* $\boldsymbol{B} = (B^1, \ldots B^k)$ *be a subsequence of* $\boldsymbol{A}$*. We assume that* $k \geq 2$*. Let* $u = (V, l, \curvearrowright, \rightsquigarrow)$ *be an justified graph over* $\boldsymbol{A}$*. The* restriction $u{\restriction_{\boldsymbol{B}}}$ of $u$ to $\boldsymbol{B}$ *is defined as follows:*

$$u{\restriction_{\boldsymbol{B}}} := (V{\restriction_{\boldsymbol{B}}},\ l{\restriction_{\boldsymbol{B}}},\ \curvearrowright{\restriction_{\boldsymbol{B}}},\ \rightsquigarrow{\restriction_{\boldsymbol{B}}})$$

*where*

- $V{\restriction_{\boldsymbol{B}}} := \{v \in V \mid l(v) \in \mathcal{M}_{\boldsymbol{B}}\}$,
- $l{\restriction_{\boldsymbol{B}}}(v) := l(v)$ *for all* $v \in V{\restriction_{\boldsymbol{B}}}$,
- $\curvearrowright{\restriction_{\boldsymbol{B}}} := \{(v, v') \in (V{\restriction_{\boldsymbol{B}}}) \times (V{\restriction_{\boldsymbol{B}}}) \mid v \curvearrowright v'\}$ *and*
- $\rightsquigarrow{\restriction_{\boldsymbol{B}}} := \{(v, v') \mid \exists i.\ (v, v') \in V^{P}_{B^i,B^{i+1}} \times V^{O}_{B^i,B^{i+1}},\ v \rightsquigarrow V^{*}_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow v'\}$.

**Definition 20 (Switching condition).** *Let* $u = (V, l, \curvearrowright, \rightsquigarrow)$ *be a justified graph over* $\boldsymbol{A} = (A^1, \ldots, A^n)$*. It satisfies the* switching condition *if:*

*If* $v \rightsquigarrow v'$*, then* $(v, v') \in V^{P}_{A^i,A^{i+1}} \times V^{O}_{A^i,A^{i+1}}$ *for some* $i \in \{1, \ldots, n-1\}$.

**Lemma 4.** *Let $u$ be a justified graph over $\boldsymbol{A}$ and $\boldsymbol{B}$ be a subsequence of $\boldsymbol{A}$. Assume that $u$ satisfies the switching condition (Definition 20). Let $(v, v') \in V^P_{B^i,B^{i+1}} \times V^O_{B^i,B^{i+1}}$ be a pair of P- and O-nodes in $u{\upharpoonright}_{\boldsymbol{B}}$. Assume that $B^i = A^j$ and $B^{i+1} = A^k$. Then $v \underset{\boldsymbol{B}}{\rightsquigarrow} v'$ in $u{\upharpoonright}_{\boldsymbol{B}}$ if and only if there exists a sequence*

$$v = v_0 \underset{u}{\rightsquigarrow} v_1 \underset{u}{\rightsquigarrow} v_2 \underset{u}{\rightsquigarrow} \dots \underset{u}{\rightsquigarrow} v_\ell \underset{u}{\rightsquigarrow} v_{\ell+1} = v'$$

*in $u$ where $v_1, v_2, \dots, v_\ell \in V_{A^{j+1},A^{j+2},\dots,A^{k-1}}$.*

**Corollary 1.** *Let $u$ be a justified graph over $\boldsymbol{A}$. Assume that $u$ satisfies the switching condition (Definition 20). Then for every $(v, v') \in V^P_{A^i,A^{i+1}} \times V^O_{A^i,A^{i+1}}$, we have $v \underset{A^i,A^{i+1}}{\rightsquigarrow} v'$ in $u{\upharpoonright}_{A^i,A^{i+1}}$ if and only if $v \underset{u}{\rightsquigarrow} v'$ in $u$.*

The next lemma shows that the composition of restrictions is again a restriction (provided that the justified graph satisfies the switching condition).

**Lemma 5.** *Let $\boldsymbol{A}$ be a tuple of arenas, $\boldsymbol{B}$ be a subsequence of $\boldsymbol{A}$ and $\boldsymbol{C}$ be a subsequence of $\boldsymbol{B}$. For every $u \in J_{\boldsymbol{A}}$ that satisfies the switching condition (Definition 20), we have*

$$(u{\upharpoonright}_{\boldsymbol{B}}){\upharpoonright}_{\boldsymbol{C}} \;=\; u{\upharpoonright}_{\boldsymbol{C}}.$$

*Proof.* Let $u = (V, l, \curvearrowright, \rightsquigarrow)$. Trivially we have

$$\begin{aligned}(V{\upharpoonright}_{\boldsymbol{B}}){\upharpoonright}_{\boldsymbol{C}} &= V{\upharpoonright}_{\boldsymbol{C}}\\ (l{\upharpoonright}_{\boldsymbol{B}}){\upharpoonright}_{\boldsymbol{C}} &= l{\upharpoonright}_{\boldsymbol{C}}\\ (\curvearrowright{\upharpoonright}_{\boldsymbol{B}}){\upharpoonright}_{\boldsymbol{C}} &= \curvearrowright{\upharpoonright}_{\boldsymbol{C}}.\end{aligned}$$

It suffices to show that $(\rightsquigarrow{\upharpoonright}_{\boldsymbol{B}}){\upharpoonright}_{\boldsymbol{C}} = \rightsquigarrow{\upharpoonright}_{\boldsymbol{C}}$. Let us write $\underset{\boldsymbol{BC}}{\rightsquigarrow}$ for the left-hand-side and $\underset{\boldsymbol{C}}{\rightsquigarrow}$ for the right-hand-side of the desired equation. We define $\underset{\boldsymbol{B}}{\rightsquigarrow} := \rightsquigarrow{\upharpoonright}_{\boldsymbol{B}}$.

Let $(v, v') \in V^P_{C^i,C^{i+1}} \times V^O_{C^i,C^{i+1}}$ for some $i$.

We first show that $v \underset{\boldsymbol{BC}}{\rightsquigarrow} v'$ implies $v \underset{\boldsymbol{C}}{\rightsquigarrow} v'$. Assume that $v \underset{\boldsymbol{BC}}{\rightsquigarrow} v'$. By definition,

$$v \underset{\boldsymbol{B}}{\rightsquigarrow} v_1 \underset{\boldsymbol{B}}{\rightsquigarrow} v_2 \underset{\boldsymbol{B}}{\rightsquigarrow} \cdots \underset{\boldsymbol{B}}{\rightsquigarrow} v_\ell \underset{\boldsymbol{B}}{\rightsquigarrow} v'$$

for some $\ell \geq 0$ and $v_1, \dots, v_l \in V_{\boldsymbol{B}\setminus\boldsymbol{C}}$. By the definition of $\underset{\boldsymbol{B}}{\rightsquigarrow}$, we have

$$v \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow v_1 \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow v_2 \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow \dots \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow v_\ell \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow v'.$$

Since $V_{\boldsymbol{A}\setminus\boldsymbol{B}} \subseteq V_{\boldsymbol{A}\setminus\boldsymbol{C}}$ and $\{v_1, \dots, v_\ell\} \subseteq V_{\boldsymbol{B}\setminus\boldsymbol{C}} \subseteq V_{\boldsymbol{A}\setminus\boldsymbol{C}}$, we have

$$v \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{C}} \rightsquigarrow v',$$

which means that $v \underset{\boldsymbol{C}}{\rightsquigarrow} v'$.

Then we show that $v \underset{\boldsymbol{C}}{\rightsquigarrow} v'$ implies $v \underset{\boldsymbol{BC}}{\rightsquigarrow} v'$. Assume that $v \underset{\boldsymbol{C}}{\rightsquigarrow} v'$. Then

$$v \rightsquigarrow v_1 \rightsquigarrow v_2 \dots \rightsquigarrow v_\ell \rightsquigarrow v'$$

for some $\ell \geq 0$ and $v_1, \ldots, v_\ell \in V_{\boldsymbol{A}\setminus\boldsymbol{C}}$. Recall that $V_{\boldsymbol{A}\setminus\boldsymbol{C}} = V_{\boldsymbol{A}\setminus\boldsymbol{B}} \uplus V_{\boldsymbol{B}\setminus\boldsymbol{C}}$. Let $v_{j_1}, v_{j_2}, \ldots, v_{j_k}$ be the substring of $v_1, \ldots, v_\ell$ consisting of nodes in $V_{\boldsymbol{B}\setminus\boldsymbol{C}}$ $(k \geq 0)$. Then

$$v \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow v_{j_1} \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow v_{j_2} \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow \ldots \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow v_{j_k} \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow v'.$$

So, if

$$\begin{aligned}
(v, v_{j_1}) &\in V^P_{B^{c_1}, B^{c_1+1}} \times V^O_{B^{c_1}, B^{c_1+1}} \\
(v_{j_1}, v_{j_2}) &\in V^P_{B^{c_2}, B^{c_2+1}} \times V^O_{B^{c_2}, B^{c_2+1}} \\
&\vdots \\
(v_{j_{k-1}}, v_{j_k}) &\in V^P_{B^{c_k}, B^{c_k+1}} \times V^O_{B^{c_k}, B^{c_k+1}} \\
(v_{j_k}, v') &\in V^P_{B^{c_{k+1}}, B^{c_{k+1}+1}} \times V^O_{B^{c_{k+1}}, B^{c_{k+1}+1}}
\end{aligned}$$

for some $c_1, \ldots, c_{k+1}$, we have

$$v \overset{\rightrightarrows}{\widehat{\boldsymbol{B}}} v_{j_1} \overset{\rightrightarrows}{\widehat{\boldsymbol{B}}} v_{j_2} \overset{\rightrightarrows}{\widehat{\boldsymbol{B}}} \cdots \overset{\rightrightarrows}{\widehat{\boldsymbol{B}}} v_{j_k} \overset{\rightrightarrows}{\widehat{\boldsymbol{B}}} v'$$

which means $v \overset{\rightrightarrows}{\widehat{\boldsymbol{B}\boldsymbol{C}}} v'$ since $v_{j_1}, \ldots, j_{j_k} \in V_{\boldsymbol{B}\setminus\boldsymbol{C}}$.

Now what remains is to prove the above assumption. Here we use the switching condition. We prove the following claim:

> Recall that $\boldsymbol{B}$ is a subsequence of $\boldsymbol{A}$. Hence, for every $p$, we have $q$ and $r$ such that $B^p = A^q$ and $B^{p+1} = A^r$. Let $w \in V$ and $w' \in V_{\boldsymbol{B}}$ and assume that $w \neq w'$. If $w \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow w'$ and $w \in V^O_{A^q} \uplus V_{A^{q+1}} \uplus \cdots \uplus V_{A^{r-1}} \uplus V^P_{A^r}$, then $w' \in V^O_{B^j, B^{j+1}}$. In particular, if $w \in V^P_{B^p, B^{p+1}}$ and $w \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow w'$ for some $w' \in V_{\boldsymbol{B}}$, then $w' \in V^O_{B^j, B^{j+1}}$.

This is proved by an easy induction on the length of $w \rightsquigarrow V^*_{\boldsymbol{A}\setminus\boldsymbol{B}} \rightsquigarrow w'$, using the switching condition of $u$. Recall that $(v, v') \in V^P_{C^i, C^{i+1}} \times V^O_{C^i, C^{i+1}}$. Since $\boldsymbol{C}$ is a subsequence of $\boldsymbol{B}$, we have $C^i = B^d$ and $C^{i+1} = B^e$ for some $d < e$. The switching condition of $u$ and the fact that $v_1, \ldots, v_\ell \notin V_{\boldsymbol{C}}$ implies $v_1, \ldots, v_\ell \in V_{B^{j+1}, \ldots, B^{k-1}}$. In particular, for every $d \in \{1, \ldots, \ell\}$, there exists $p$ such that $v_d \in V^P_{B^p, B^{p+1}}$. Hence we have

$$\begin{aligned}
(v, v_{j_1}) &\in V^P_{B^{c_1}, B^{c_1+1}} \times V^O_{B^{c_1}, B^{c_1+1}} \\
(v_{j_1}, v_{j_2}) &\in V^P_{B^{c_2}, B^{c_2+1}} \times V^O_{B^{c_2}, B^{c_2+1}} \\
&\vdots \\
(v_{j_{k-1}}, v_{j_k}) &\in V^P_{B^{c_k}, B^{c_k+1}} \times V^O_{B^{c_k}, B^{c_k+1}} \\
(v_{j_k}, v') &\in V^P_{B^{c_{k+1}}, B^{c_{k+1}+1}} \times V^O_{B^{c_{k+1}}, B^{c_{k+1}+1}}
\end{aligned}$$

as required. $\square$

**Definition 21 (Interaction graph).** *Let $\boldsymbol{A} = (A^1, \dots, A^n)$ be a tuple of arenas. A justified graph $u = (V, l, \curvearrowright, \rightsquigarrow) \in J_{\boldsymbol{A}}$ is an* interaction graph *if it satisfies the following conditions.*

**(I1)** *If $v \rightsquigarrow v'$, then $(v, v') \in V^P_{A^i,A^{i+1}} \times V^O_{A^i,A^{i+1}}$ for some $i \in \{1, \dots, n-1\}$. (The switching condition, Definition 20.)*
**(I2)** *$u\restriction_{A^i,A^{i+1}}$ is a play for every $i \in \{1, \dots, n-1\}$.*

*Condition (I1) is called the* switching condition. *We write* $\mathrm{Int}(\boldsymbol{A})$ *for the set of interaction graphs over $\boldsymbol{A}$.*

The above notion of interaction graphs is a generalisation of the notions of plays (Definition 4) and interaction graphs (Definition 9). In fact $\mathrm{Int}(A, B) = P_{A,B}$ and $\mathrm{Int}(A, B, C)$ in Definition 9 coincides with that of Definition 21.

**Lemma 6.** *Let $\boldsymbol{A} = (A^1, \dots, A^n)$ be a tuple of arenas and $\boldsymbol{B}$ be a subsequence of $\boldsymbol{A}$ of length greater than 1. If $u \in \mathrm{Int}(\boldsymbol{A})$, then $u\restriction_{\boldsymbol{B}} \in \mathrm{Int}(\boldsymbol{B})$. Especially, if $\boldsymbol{B}$ is a pair then $u\restriction_{\boldsymbol{B}}$ is a play.*

*Proof.* $u\restriction_{\boldsymbol{B}}$ satisfies the switching condition by definition. Thanks to Lemma 5, it suffices to show the case where $\boldsymbol{B} = (A^i, A^j)$ with $i < j$. For general case, we have $(u\restriction_{\boldsymbol{B}})\restriction_{B^\ell,B^{\ell+1}} = u\restriction_{B^\ell,B^{\ell+1}} = u\restriction_{A^i,A^j} \in P_{A^i,A^j}$ for appropriate $i$ and $j$.

Let $u = (V, l, \curvearrowright, \rightsquigarrow)$ be a justified graph over $\boldsymbol{A}$ and $\boldsymbol{B} = (A^i, A^j)$ $(i < j)$ be a subsequence of $\boldsymbol{A}$. Let $u\restriction_{\boldsymbol{B}} = (V_{\boldsymbol{B}}, l_{\boldsymbol{B}}, \curvearrowright_{\boldsymbol{B}}, \rightsquigarrow_{\boldsymbol{B}})$ be the restriction of $u$ to $\boldsymbol{B}$. We prove that $u\restriction_{\boldsymbol{B}}$ satisfies (P1), (P2) and (P3).

Condition (P1) follows from the definition of the restriction.

We prove Condition (P2). Let $v \in V^P_{\boldsymbol{B}}$ and $v' \in V^O_{\boldsymbol{B}}$ and assume that $v \rightarrow_{\boldsymbol{B}}^{+} v'$, i.e.,

$$v = v_0 \rightarrow_{\boldsymbol{B}} v_1 \rightarrow_{\boldsymbol{B}} v_2 \rightarrow_{\boldsymbol{B}} \cdots \rightarrow_{\boldsymbol{B}} v_\ell \rightarrow_{\boldsymbol{B}} v_{\ell+1} = v'.$$

We prove that $v \rightsquigarrow_{\boldsymbol{B}} v'$ by induction on the length of the above sequence.

- Case $v \rightsquigarrow_{\boldsymbol{B}} v'$: This is the claim itself.
- Case $v \curvearrowright_{\boldsymbol{B}} v'$: Since $v \in V^P_{\boldsymbol{B}}$, there are two cases (recall that $\boldsymbol{B} = (A^i, A^j)$).
  - $v \in V^O_{A^i}$: Then $v \in V^P_{A^i,A^{i+1}}$. Since $v' \in V^O_{\boldsymbol{B}}$ and $v \curvearrowright_{\boldsymbol{B}} v'$, we have $v' \in V^P_{A^i}$ (recall that a justification pointer connects only moves in the same arena) and thus $v' \in V^O_{A^i,A^{i+1}}$. Because $u$ is an interaction graph, we have $u\restriction_{A^i,A^{i+1}} \in P_{A^i,A^{i+1}}$. By Condition (P2) for $u\restriction_{A^i,A^{i+1}}$, we have $v \rightsquigarrow_{A^i,A^{i+1}} v'$ in $u\restriction_{A^i,A^{i+1}}$. Then, by Corollary 1, we have $v \rightsquigarrow v'$ in $u$. Hence $v \rightsquigarrow_{\boldsymbol{B}} v'$.
  - $v \in V^P_{A^j}$: Then $v \in V^P_{A^{j-1},A^j}$ and $v' \in V^O_{A^{j-1},A^j}$. By the same argument as above, we have $v \rightsquigarrow_{A^{j-1},A^j} v'$ in $u\restriction_{A^{j-1},A^j}$. Then, by Corollary 1, we have $v \rightsquigarrow v'$ in $u$. Hence $v \rightsquigarrow_{\boldsymbol{B}} v'$.
- Case $v \rightsquigarrow_{\boldsymbol{B}} v_1 \rightarrow_{\boldsymbol{B}}^{+} v'$: Then $v_1 \in V^O_{\boldsymbol{B}}$ and thus the edge $v_1 \rightarrow_{\boldsymbol{B}} v_2$ is a justification pointer $v_1 \curvearrowright_{\boldsymbol{B}} v_2$. There are two cases.

- Case $v_2 \in V^O_{\boldsymbol{B}}$: Since $v \underset{\boldsymbol{B}}{\rightsquigarrow} v_1 \underset{\boldsymbol{B}}{\curvearrowright} v_2$, we have

$$v \rightsquigarrow w_1 \rightsquigarrow \ldots \rightsquigarrow w_k \rightsquigarrow v_1 \curvearrowright v_2$$

in $u$, where $w_1, \ldots, w_k \in V_{\boldsymbol{A} \setminus \boldsymbol{B}}$. Let

$$\boldsymbol{C} = \begin{cases} (A^i, A^{i+1}) & (\text{if } v_2 \in V_{A^i}) \\ (A^{j-1}, A^j) & (\text{if } v_2 \in V_{A^j}). \end{cases}$$

By the switching condition, $(w_k, v_1) \in V^P_{\boldsymbol{C}} \times V^O_{\boldsymbol{C}}$. We have $w_k \underset{\boldsymbol{C}}{\rightsquigarrow} v_1$ in $u{\restriction}_{\boldsymbol{C}}$. Furthermore, because a justification pointer connects moves in the same component, we have $v_2 \in V_{\boldsymbol{C}}$. So

$$w_k \underset{\boldsymbol{C}}{\rightsquigarrow} v_1 \underset{\boldsymbol{C}}{\curvearrowright} v_2$$

in $u{\restriction}_{\boldsymbol{C}}$. Then we have $w_k \underset{\boldsymbol{C}}{\rightsquigarrow} v_2$ because of Condition (P2) for $u{\restriction}_{\boldsymbol{C}}$, which is a play by the assumption that $u$ is an interaction graph. By Corollary 1, we have $w_k \rightsquigarrow v_2$ in $u$ and thus

$$v \rightsquigarrow w_1 \rightsquigarrow \ldots \rightsquigarrow w_k \rightsquigarrow v_2,$$

which means that $v \underset{\boldsymbol{B}}{\rightsquigarrow} v_2$ since $w_1, \ldots, w_k \in V_{\boldsymbol{A} \setminus \boldsymbol{B}}$. Now we have

$$v = v_0 \underset{\boldsymbol{B}}{\to} v_2 \underset{\boldsymbol{B}}{\to} v_3 \underset{\boldsymbol{B}}{\to} \cdots \underset{\boldsymbol{B}}{\to} v_\ell \underset{\boldsymbol{B}}{\to} v_{\ell+1} = v'.$$

Since the length of this sequence is smaller than the original sequence, by the induction hypothesis, we have $v \underset{\boldsymbol{B}}{\rightsquigarrow} v'$.
- Case $v_2 \in V^P_{\boldsymbol{B}}$: Then $v_2 \neq v'$ since $v' \in V^O_{\boldsymbol{B}}$. By the induction hypothesis, we have $v_2 \underset{\boldsymbol{B}}{\rightsquigarrow} v'$. Hence

$$v \rightsquigarrow w_1 \rightsquigarrow \ldots \rightsquigarrow w_k \rightsquigarrow v_1 \curvearrowright v_2 \rightsquigarrow z_1 \rightsquigarrow \ldots \rightsquigarrow z_m \rightsquigarrow v'$$

where $w_1, \ldots, w_k, z_1, \ldots, z_m \in V_{\boldsymbol{A} \setminus \boldsymbol{B}}$. Let

$$\boldsymbol{C} = \begin{cases} (A^i, A^{i+1}) & (\text{if } v_2 \in V_{A^i}) \\ (A^{j-1}, A^j) & (\text{if } v_2 \in V_{A^j}). \end{cases}$$

By the switching condition, $(w_k, v_1) \in V^P_{\boldsymbol{C}} \times V^O_{\boldsymbol{C}}$. Furthermore, because a justification pointer connects moves in the same component, we have $v_2 \in V^P_{\boldsymbol{C}}$ (here the polarity comes from the assumption that $v_2 \in V^P_{\boldsymbol{B}}$) and $(v_2, z_1) \in V^P_{\boldsymbol{C}} \times V^O_{\boldsymbol{C}}$. Then we have

$$w_k \underset{\boldsymbol{C}}{\rightsquigarrow} v_1 \underset{\boldsymbol{C}}{\curvearrowright} v_2 \underset{\boldsymbol{C}}{\rightsquigarrow} z_1$$

in $u{\restriction}_{\boldsymbol{C}}$, which is a play since $u$ is an interaction graph. By Condition (P2) for $u{\restriction}_{\boldsymbol{C}}$, we have $w_k \underset{\boldsymbol{C}}{\rightsquigarrow} z_1$. By Corollary 1, we have $w_k \rightsquigarrow z_1$ in $u$. Hence we have

$$v \rightsquigarrow w_1 \rightsquigarrow \ldots \rightsquigarrow w_k \rightsquigarrow z_1 \rightsquigarrow \ldots \rightsquigarrow z_m \rightsquigarrow v'$$

which implies $v \underset{\boldsymbol{B}}{\rightsquigarrow} v'$ since $w_1, \ldots, w_k, z_1, \ldots, z_m \in V_{\boldsymbol{A} \setminus \boldsymbol{B}}$.

– Case $v \curvearrowright_{\boldsymbol{B}} v_1 \to_{\boldsymbol{B}}^{+} v'$: If $v_1 \in V_{\boldsymbol{B}}^{O}$, then the edge $v_1 \to_{\boldsymbol{B}} v_2$ is a justification pointer $v_1 \to_{\boldsymbol{B}} v_2$. By iterating this argument, we have either

$$v \curvearrowright_{\boldsymbol{B}} v_1 \curvearrowright_{\boldsymbol{B}} \cdots \curvearrowright_{\boldsymbol{B}} v_k \to_{\boldsymbol{B}}^{+} v'$$

for some $v_k \in V_{\boldsymbol{B}}^{P}$ or

$$v \curvearrowright_{\boldsymbol{B}} v_1 \curvearrowright_{\boldsymbol{B}} \cdots \curvearrowright_{\boldsymbol{B}} v'.$$

Let

$$\boldsymbol{C} = \begin{cases} (A^i, A^{i+1}) & (\text{if } v \in V_{A^i}) \\ (A^{j-1}, A^j) & (\text{if } v \in V_{A^j}). \end{cases}$$

- Case $v \curvearrowright_{\boldsymbol{B}} v_1 \curvearrowright_{\boldsymbol{B}} \ldots \curvearrowright_{\boldsymbol{B}} v_k \to_{\boldsymbol{B}}^{+} v'$ with $v_k \in V_{\boldsymbol{B}}^{P}$: By the induction hypothesis, we have $v_k \rightsquigarrow_{\boldsymbol{B}} v'$ in $u\restriction_{\boldsymbol{B}}$. Hence we have

$$v \curvearrowright v_1 \curvearrowright \ldots \curvearrowright v_k \rightsquigarrow w_1 \rightsquigarrow w_2 \rightsquigarrow \ldots \rightsquigarrow w_m \rightsquigarrow v'$$

  where $w_1, \ldots, w_m \in V_{\boldsymbol{A} \setminus \boldsymbol{B}}$. Since a justification pointer connects moves in the same component, we have $v, v_1, \ldots, v_k \in V_{\boldsymbol{C}}$. Since $v_k \in V_{\boldsymbol{B}}^{P}$, we have $v_k \in V_{\boldsymbol{C}}^{P}$. Hence $(v_k, w_1) \in V_{\boldsymbol{C}}^{P} \times V_{\boldsymbol{C}}^{O}$ by the switching condition. Now we have

$$v \curvearrowright_{\boldsymbol{C}} v_1 \curvearrowright_{\boldsymbol{C}} \cdots \curvearrowright_{\boldsymbol{C}} v_k \rightsquigarrow_{\boldsymbol{C}} w_1$$

  in $u\restriction_{\boldsymbol{C}}$, which is a play since $u$ is an interaction graph. Condition (P2) for $u\restriction_{\boldsymbol{C}}$ implies $v \rightsquigarrow_{\boldsymbol{C}} w_1$. By Corollary 1, we have $v \rightsquigarrow w_1$ in $u$. Hence

$$v \rightsquigarrow w_1 \rightsquigarrow w_2 \rightsquigarrow \ldots \rightsquigarrow w_m \rightsquigarrow v'$$

  which implies $v \rightsquigarrow_{\boldsymbol{B}} v'$ since $w_1, \ldots, w_m \in V_{\boldsymbol{A} \setminus \boldsymbol{B}}$.
- Case $v \curvearrowright_{\boldsymbol{B}} v_1 \curvearrowright_{\boldsymbol{B}} \cdots \curvearrowright_{\boldsymbol{B}} v'$: Then by the definition of the restriction,

$$v \curvearrowright v_1 \curvearrowright \ldots \curvearrowright v'$$

  in $u$. Then $v, v_1, \ldots, v_\ell, v' \in V_{\boldsymbol{C}}$ since a justification pointer connects moves in the same component. Thus

$$v \curvearrowright_{\boldsymbol{C}} v_1 \curvearrowright_{\boldsymbol{C}} \cdots \curvearrowright_{\boldsymbol{C}} v'$$

  in $u\restriction_{\boldsymbol{C}}$, which is a play. Since $(v, v') \in V_{\boldsymbol{C}}^{P} \times V_{\boldsymbol{C}}^{O}$, Condition (P2) implies $v \rightsquigarrow_{\boldsymbol{C}} v'$. By Corollary 1, we have $v \rightsquigarrow v'$ in $u$. Hence $v \rightsquigarrow_{\boldsymbol{B}} v'$ in $u\restriction_{\boldsymbol{B}}$.

We prove Condition (P3). Let $\boldsymbol{C} = (A^i, A^{i+1}, \ldots, A^{j-1}, A^j)$. Since $u\restriction_{A^k, A^{k+1}}$ is a play for every $k \in \{i, i+1, \ldots, j-1\}$, we have the following claim:

For every node $w \in V_{\boldsymbol{C}}$ except for $w \in V_{\boldsymbol{B}}^{P}$, there exists a node $w' \in (V_{\boldsymbol{C}} \setminus V_{\boldsymbol{B}}^{O})$ such that $w' \rightsquigarrow w$ in $u$.

To prove this claim, we use Condition (P3) for $u{\restriction}_{A^k,A^{k+1}}$, the switching condition of $u$, and Corollary 1. Given $v \in V^O_{\boldsymbol{B}}$, consider a family of sequences of the form

$$w_n \rightsquigarrow w_{n-1} \rightsquigarrow \ldots \rightsquigarrow w_1 \rightsquigarrow v$$

where $w_1, \ldots, w_n \in (V_{\boldsymbol{C}} \setminus V^O_{\boldsymbol{B}})$. Since an interaction graph has only finitely many nodes and it is acyclic, there exists a maximal chain, in which $\neg\exists w_{n+1} \in (V_{\boldsymbol{C}} \setminus V^O_{\boldsymbol{B}}).\ w_{n+1} \rightsquigarrow w_n$. By the above claim, we have $w_n \in V^P_{\boldsymbol{B}}$, in particular $\{w_1, \ldots, w_n\}$ contains at least one $V^P_{\boldsymbol{B}}$ node. Hence there exists $k \leq n$ such that $w_k \in V^P_{\boldsymbol{B}}$ and $\{w_1, \ldots, w_{k-1}\} \subseteq ((V_{\boldsymbol{C}} \setminus V^O_{\boldsymbol{B}}) \setminus V^P_{\boldsymbol{B}}) = (V_{\boldsymbol{C}} \setminus V_{\boldsymbol{B}})$. So we have a sequence

$$w_k \rightsquigarrow w_{k-1} \rightsquigarrow \ldots \rightsquigarrow w_1 \rightsquigarrow v$$

such that $(w_k, v) \in V^P_{\boldsymbol{B}} \times V^O_{\boldsymbol{B}}$ and $w_{k-1}, \ldots, w_1 \in V_{\boldsymbol{C} \setminus \boldsymbol{B}} \subseteq V_{\boldsymbol{A} \setminus \boldsymbol{B}}$. Hence $w_k \underset{\boldsymbol{B}}{\rightsquigarrow} v$ in $u{\restriction}_{\boldsymbol{B}}$. □

### B.4 Proof of Theorem 1

**Proposition 1.** *$\tau \circ \sigma$ is a strategy for every strategies $\sigma : A \to B$ and $\tau : B \to C$.*

*Proof.* We show that $\tau \circ \sigma$ is prefixed closed (Condition (S1) in Definition 6). Let $s \in (\tau \circ \sigma)$ and $s' \sqsubseteq s$. By the definition of the composition, there exists an interaction graph $u \in \mathrm{Int}(A, B, C)$ such that

1. $u{\restriction}_{A,B} \in \sigma$,
2. $u{\restriction}_{B,C} \in \tau$, and
3. $u{\restriction}_{A,C} = s$.

Let $u = (V, l, \curvearrowright, \rightsquigarrow)$. By the definition of $s' \sqsubseteq s$, we have a subset $W_s \subseteq V_s$ of nodes of $s$ such that $s' = s[W_s]$ (see Definition 5). Recall that $V_s \subseteq V$ by the definition of $u{\restriction}_{A,C}$. Let $W \subseteq V_u$ be the subset of nodes of $u$ defined by

$$W := \{v \in V \mid \exists v' \in W_s.\ v' \underset{u}{\rightarrow}^* v\}.$$

Consider $u[W] := (W_u, l_W, \underset{W}{\curvearrowright}, \underset{W}{\rightsquigarrow})$ (see Definition 18). Obviously it is a justified graph and satisfies the switching condition, since $u$ does. It is also not difficult to see that $W \cap V_{A,B}$ and $W \cap V_{B,C}$ satisfy the requirement in Definition 5 with respect to $u{\restriction}_{A,B}$ and $u{\restriction}_{B,C}$, respectively (Condition (2) in Definition 5 follows from that for $W_s$ with respect to $s$ and the switching condition of $u$). Hence $u[W]$ is an interaction graph.

Then we have

$$(u{\restriction}_{A,B})[W \cap V_{A,B}] = (u[W]){\restriction}_{A,B}$$

and

$$(u{\restriction}_{B,C})[W \cap V_{B,C}] = (u[W]){\restriction}_{B,C}.$$

This shows that $(u[W]){\restriction}_{A,B} \sqsubseteq u{\restriction}_{A,B}$ and $(u[W])({\restriction}_{B,C}) \sqsubseteq u{\restriction}_{B,C}$, and thus

1. $(u[W]){\restriction}_{A,B} \in \sigma$, and
2. $(u[W]){\restriction}_{B,C} \in \tau$.

So $(u[W])\restriction_{A,C} \in (\tau \circ \sigma)$. It is not difficult to show that $(u[W])\restriction_{A,C} = s[W_s]$. $\square$

The associativity of the composition is shown using the zipping lemma.

**Lemma 7.** *Given $u \in \mathrm{Int}(A, B, D)$ and $v \in \mathrm{Int}(B, C, D)$ such that $u\restriction_{B,D} = v\restriction_{B,D}$, there exists $w \in \mathrm{Int}(A, B, C, D)$ such that $w\restriction_{A,B,D} = u$ and $w\restriction_{B,C,D} = v$. Similarly, if $u \in \mathrm{Int}(A, C, D)$ and $v \in \mathrm{Int}(A, B, C)$ such that $u\restriction_{A,C} = v\restriction_{A,C}$, then there exists $w \in \mathrm{Int}(A, B, C, D)$ such that $w\restriction_{A,C,D} = u$ and $w\restriction_{A,B,C} = v$.*

*Proof.* We prove the former. The latter can be proved by the same way. The idea is to "glue" the two interaction graphs together overlapping the moves in $\mathcal{M}_{B,D}$.

Let $u = (V_u, l_u, \underset{u}{\curvearrowright}, \underset{u}{\rightsquigarrow})$ and $w = (V_w, l_w, \underset{w}{\curvearrowright}, \underset{w}{\rightsquigarrow})$ be interaction graphs over $(A, B, D)$ and $(B, C, D)$. We can assume without loss of generality that $V_u \cap V_w = V_u\restriction_{B,C} = V_w\restriction_{B,C}$. We define $z = (V_z, l_z, \underset{z}{\curvearrowright}, \underset{z}{\rightsquigarrow})$ by

$$
\begin{aligned}
V_z &:= V_u \cup V_w \\
l_z(v) &:= \begin{cases} l_u(v) & (\text{if } v \in V_u) \\ l_w(v) & (\text{otherwise}) \end{cases} \\
\underset{z}{\curvearrowright} &:= (\underset{u}{\curvearrowright}) \cup (\underset{w}{\curvearrowright}) \\
\underset{z}{\rightsquigarrow} &:= \{(v, v') \in (V_u)^P_{A,B} \times (V_u)^O_{A,B} \mid v \underset{u}{\rightsquigarrow} v'\} \cup (\underset{w}{\rightsquigarrow}).
\end{aligned}
$$

Note that the causal relation in $(B, D)$ component of $u$ is not added to $\underset{z}{\rightsquigarrow}$ since it is obtained from the causal relation of $w$ by hiding $C$. It is not difficult to see that $z$ satisfies the requirements. $\square$

**Proposition 2.** *Let $\sigma : A \to B$, $\tau : B \to C$ and $\delta : C \to D$ be strategies. We have $(\delta \circ \tau) \circ \sigma = \delta \circ (\tau \circ \sigma)$.*

*Proof.* We show that $(\delta \circ \tau) \circ \sigma \subseteq \delta \circ (\tau \circ \sigma)$. Similar arguments apply to the opposite inclusion.

Let $s \in (\delta \circ \tau) \circ \sigma$ and $u$ be a witness of $s$, i.e. an interaction graph $u \in \mathrm{Int}(A, B, D)$ such that

- $u\restriction_{A,B} \in \sigma$,
- $u\restriction_{B,D} \in (\delta \circ \tau)$ and
- $u\restriction_{A,D} = s$.

Since $u\restriction_{B,D} \in \delta \circ \tau$, there exists $v$ a witness of $u\restriction_{B,D}$, i.e. an interaction graph $v \in \mathrm{Int}(B, C, D)$ such that

- $v\restriction_{B,C} \in \tau$,
- $v\restriction_{C,D} \in \delta$ and
- $v\restriction_{B,D} = u\restriction_{B,D}$.

By Lemma 7, there exists $w \in \mathrm{Int}(A, B, C, D)$ such that $w\restriction_{A,B,D} = u$ and $w\restriction_{B,C,D} = v$.

We show that $w\restriction_{A,C,D}$ is a witness of $s$ as a member of $\delta \circ (\tau \circ \sigma)$. Using Lemma 5, we have

$$\begin{aligned}
(w\restriction_{A,C,D})\restriction_{A,D} &= (w\restriction_{A,B,D})\restriction_{A,D} \\
&= u\restriction_{A,D} \\
&= s
\end{aligned}$$

and

$$\begin{aligned}
(w\restriction_{A,C,D})\restriction_{C,D} &= (w\restriction_{B,C,D})\restriction_{C,D} \\
&= v\restriction_{C,D} \in \delta
\end{aligned}$$

Next we need to prove that $(w\restriction_{A,C,D})\restriction_{A,C} \in \tau \circ \sigma$, meaning that we have to find a witness for $(w\restriction_{A,C,D})\restriction_{A,C}$ as a member of $\tau \circ \sigma$. We claim that $w\restriction_{A,B,C}$ is the witness of $(w\restriction_{A,C,D})\restriction_{A,C}$. This is proved by

$$\begin{aligned}
(w\restriction_{A,C,D})\restriction_{A,C} &= (w\restriction_{A,B,D})\restriction_{A,C} \\
(w\restriction_{A,B,C})\restriction_{A,B} &= (w\restriction_{A,B,D})\restriction_{A,B} \\
&= u\restriction_{A,B} \in \sigma
\end{aligned}$$

and

$$\begin{aligned}
(w\restriction_{A,B,C})\restriction_{B,C} &= (w\restriction_{B,C,D})\restriction_{B,C} \\
&= v\restriction_{B,C} \in \tau
\end{aligned}$$

Thus $s \in \delta \circ (\tau \circ \sigma)$.

Because $s$ is arbitrary, we have $(\delta \circ \tau) \circ \sigma \subseteq \delta \circ (\tau \circ \sigma)$. □

### B.5 On the copycat strategy

Here we give the formal definition of the copycat strategy that was illustrated in Section 3.3.

**Definition 22 (Copycat graph, copycat play).** *Let $A$ be a negative arena. We write $(A^1, A^2)$ for the arena pair consisting of two copies of $A$, whose move set is $\mathcal{M}_A + \mathcal{M}_A = \{(i, m) \mid i \in \{1, 2\},\ m \in \mathcal{M}_A\}$. Given $m \in \mathcal{M}_{A^1,A^2}$, we define $\overline{m}$ by*

$$\overline{(1, m_0)} := (2, m_0) \qquad \overline{(2, m_0)} := (1, m_0).$$

*Let $s = (V, l, \curvearrowright, \rightsquigarrow)$ be a justified graph over $(A^1, A^2)$ and $V^i := \{v \in V \mid \exists m.\ l_s(v) = (i, m)\}$ be the set of nodes of the $i$-th component ($i = 1, 2$). We say that $s$ is a* copycat graph *if there exists a bijection $\overline{v} : V \to V$ that satisfies the following conditions:*

- *$v \in V^1$ implies $\overline{v} \in V^2$, and $v \in V^2$ implies $\overline{v} \in V^1$,*
- *$l(\overline{v}) = \overline{l(v)}$ for every $v \in V$,*

- $v_1 \curvearrowright v_2$ *iff* $\overline{v_1} \curvearrowright \overline{v_2}$*, and*
- $v_1 \rightsquigarrow v_2$ *iff* $(v_1, v_2) \in V^P \times V^O$ *and* $v_1 = \overline{v_2}$*.*

*A copycat graph is not necessarily a play because of Condition (P2). Given a copycat graph $s$, the corresponding* copycat play *is* $(V, l, \curvearrowright, (\rightsquigarrow) \cup \{(v, v') \in V^P \times V^O \mid v \to^* v'\})$*.*

The notation $\overline{v}$ is used to represent the copy of $v$ throughout this paper in different contexts. For an arena pair $(A^1, A^2)$, we write $\mathbf{cc}(s)$ to refer to the copycat graph over $(A^1, A^2)$ such that $\mathbf{cc}(s)\restriction_{A^2} = s$.

We write $\mathbb{C}_A$ for the set of copycat graphs and $\mathrm{id}_A$ for the set of copycat plays. We call $\mathrm{id}_A$ the *copycat strategy.*

**Proposition 3.** *Let $s = (V, l, \curvearrowright, \rightsquigarrow)$ be a copycat play with the bijection $\overline{\cdot}$. Let $(v_1, v_2) \in V^P \times V^O$. Then $v_1 \rightsquigarrow v_2$ if and only if $v_1 \curvearrowright^* \overline{v_2}$ or $v_1 \curvearrowright^* v_2$.*

*Proof.* By the definition of a copycat play, we have a copycat graph $s_0 = (V, l, \curvearrowright, \underset{0}{\rightsquigarrow})$ such that $v_1 \underset{0}{\rightsquigarrow} v_2$ if and only if $(v_1, v_2) \in V^P \times V^O$ and $v_1 = \overline{v_2}$.

Assume that $v_1 \curvearrowright^* \overline{v_2}$. We have $\overline{v_2} \in V^P$ since $v_2 \in V^O$. Hence we have $\overline{v_2} \underset{0}{\rightsquigarrow} v_2$ and thus $v_1 \curvearrowright^* \overline{v_2} \underset{0}{\rightsquigarrow} v_2$. So, by definition, we have $v_1 \rightsquigarrow v_2$.

Assume that $v_1 \curvearrowright^* v_2$. By definition, we have $v_1 \rightsquigarrow v_2$.

Assume that $v_1 \rightsquigarrow v_2$. By the definition of $\rightsquigarrow$, we have $(v_1, v_2) \in V^P \times V^O$ and $v_1 \underset{0}{\to}^* v_2$, where $(\underset{0}{\to}) = (\curvearrowright) \cup (\underset{0}{\rightsquigarrow})$. Let

$$v_1 = w_1 \underset{0}{\to} w_2 \underset{0}{\to} \dots \underset{0}{\to} w_{n-1} \underset{0}{\to} w_n = v_2.$$

If this sequence has more than one causal edges, we rewrite a subsequence

$$w_i \underset{0}{\rightsquigarrow} w_{i+1} \curvearrowright w_{i+2} \curvearrowright \dots \curvearrowright w_{j-2} \curvearrowright w_{j-1} \underset{0}{\rightsquigarrow} w_j$$

to

$$w_i \curvearrowright \overline{w_{i+1}} \curvearrowright \dots \curvearrowright \overline{w_{j-2}} \curvearrowright w_j.$$

This is possible because

- $w_i \underset{0}{\rightsquigarrow} w_{i+1}$ implies $w_{i+1} = \overline{w_i}$,
- $\overline{w_i} \curvearrowright w_{i+2}$ implies $w_i = \overline{\overline{w_i}} \curvearrowright \overline{w_{i+2}}$,
- $w_k \curvearrowright w_{k+1}$ implies $\overline{w_k} \curvearrowright \overline{w_{k+1}}$, for every $k \in \{i+2, \dots, j-3\}$,
- $w_{j-2} \curvearrowright w_{j-1} \underset{0}{\rightsquigarrow} w_j$ implies $w_{j-1} = \overline{w_j}$ and $\overline{w_{j-2}} \curvearrowright \overline{\overline{w_j}} = w_j$.

So we can assume without loss of generality that the sequence $v_1 \underset{0}{\to}^* v_2$ contains at most one causal edge. If it has no causal edge, we have $v_1 \curvearrowright^* v_2$. Otherwise, we have

$$v_1 = w_1 \curvearrowright w_2 \curvearrowright \dots \curvearrowright w_i \underset{0}{\rightsquigarrow} w_{i+1} \curvearrowright w_{i+2} \curvearrowright \dots \curvearrowright w_n = v_2 \in V^O.$$

Then we have

$$v_1 = w_1 \curvearrowright \dots \curvearrowright w_i \curvearrowright \overline{w_{i+2}} \curvearrowright \overline{w_{i+3}} \curvearrowright \dots \curvearrowright \overline{w_n} \underset{0}{\rightsquigarrow} w_n$$

because

- $w_i \underset{0}{\rightsquigarrow} w_{i+1}$ implies $w_{i+1} = \overline{w_i}$,
- $\overline{w_i} \curvearrowright w_{i+2}$ implies $w_i = \overline{\overline{w_i}} \curvearrowright \overline{w_{i+2}}$,
- $w_k \curvearrowright w_{k+1}$ implies $\overline{w_k} \curvearrowright \overline{w_{k+1}}$ for every $k \in \{i+2, \dots, n-1\}$, and
- $\overline{w_n} \underset{0}{\rightsquigarrow} w_n$ since $w_n \in V^O$.

So we have $v_1 \curvearrowright^* \overline{v_2}$. $\square$

**Lemma 8.** *For every interaction graph $u \in \mathrm{Int}(A, B^1, B^2)$ (where we use superscripts to distinguish different copies of $B$), if $u{\restriction}_{B^1,B^2}$ is a copycat play, then $u{\restriction}_{A,B^1} = u{\restriction}_{A,B^2}$. Similarly, for every interaction graph $u \in \mathrm{Int}(A^1, A^2, B)$, if $u{\restriction}_{A^1,A^2}$ is a copycat play, then $u{\restriction}_{A^1,B} = u{\restriction}_{A^2,B}$.*

*Proof.* Let $u = (V, l, \curvearrowright, \rightsquigarrow)$ be an interaction graph over $(A, B^1, B^2)$. Given a $B^1$-move $m$, we write $\overline{m}$ for the copy of $m$ in the $B^2$ component. Similarly $\overline{m}$ for a $B^2$-move $m$ is the copy in $B^1$ component.

Assume that $u{\restriction}_{B^1,B^2}$ is a copycat play. Then we have a bijection $\overline{\cdot} : V_{B^1,B^2} \to V_{B^1,B^2}$ such that

- $v \in V_{B^i}$ implies $\overline{v} \in V_{B^{3-i}}$,
- $l(\overline{v}) = \overline{l(v)}$ for every $v \in V_{B^1,B^2}$,
- $v_1 \curvearrowright v_2$ iff $\overline{v_1} \curvearrowright \overline{v_2}$, for every $v_1, v_2 \in V_{B^1,B^2}$, and
- for every $(v_1, v_2) \in V^P_{B^1,B^2} \times V^O_{B^1,B^2}$, we have $v_1 \rightsquigarrow v_2$ iff $v_1 \curvearrowright^* v_2$ or $v_1 \curvearrowright^* \overline{v_2}$ (Proposition 3 and Corollary 1).

Let $s_1 = u{\restriction}_{A,B^1}$ and $s_2 = u{\restriction}_{A,B^2}$. Assume that $s_i = (V_i, l_i, \underset{i}{\curvearrowright}, \underset{i}{\rightsquigarrow})$ for $i = 1, 2$. Let $f : V_1 \to V_2$ be the function on nodes defined by

$$f(v) = \begin{cases} v & (\text{if } v \in V_A) \\ \overline{v} & (\text{if } v \in V_{B^1}). \end{cases}$$

Then $f$ is a bijection and preserves moves (when we identify $m \in \mathcal{M}_{B^1}$ with $\overline{m} \in \mathcal{M}_{B^2}$) and justification pointers.

We prove that $v_1 \underset{1}{\rightsquigarrow} v_2$ if and only if $f(v_1) \underset{2}{\rightsquigarrow} f(v_2)$.

Assume that $v_1 \underset{1}{\rightsquigarrow} v_2$. By Corollary 1, we have $v_1 \rightsquigarrow v_2$. There are four cases:

- $(v_1, v_2) \in V^O_A \times V^P_A$: Then $f(v_1) = v_1 \rightsquigarrow v_2 = f(v_2)$ and thus $f(v_1) \underset{2}{\rightsquigarrow} f(v_2)$.
- $(v_1, v_2) \in V^O_A \times V^O_{B^1}$: Then $v_2 \in V^P_{B^1,B^2}$. So $f(v_1) = v_1 \rightsquigarrow v_2 \rightsquigarrow \overline{v_2} = f(v_2)$ and thus $f(v_1) \underset{2}{\rightsquigarrow} f(v_2)$.
- $(v_1, v_2) \in V^P_{B^1} \times V^P_A$: Then $v_1 \in V^O_{B^1,B^2}$. So $f(v_1) = \overline{v_1} \rightsquigarrow v_1 \rightsquigarrow v_2 = f(v_2)$ and thus $f(v_1) \underset{2}{\rightsquigarrow} f(v_2)$.
- $(v_1, v_2) \in V^P_{B^1} \times V^O_{B^1}$: Then $v_1 \in V^O_{B^1,B^2}$ and $v_2 \in V^P_{B^1,B^2}$. So $f(v_1) = \overline{v_1} \rightsquigarrow v_1 \rightsquigarrow v_2 \rightsquigarrow \overline{v_2} = f(v_2)$ and thus $f(v_1) \underset{2}{\rightsquigarrow} f(v_2)$.

Assume that $f(v_1) \underset{2}{\rightsquigarrow} f(v_2)$. This means that

$$f(v_1) \rightsquigarrow w_1 \rightsquigarrow \dots \rightsquigarrow w_n \rightsquigarrow w_{n+1} = f(v_2)$$

for some $w_1, \dots, w_n \in V_{B^1}$. We prove the claim by induction on the length of the sequence.

- Case $v_1 \in V_A^O$: Then $f(v_1) = v_1$ and $v_1 \rightsquigarrow w_1$. If $w_1 = f(v_2)$, then $v_2 \in V_A^P$ and $v_1 \rightsquigarrow v_2$. Hence $v_1 \underset{1}{\rightarrowtail} v_2$. Otherwise we have $w_1 \in V_{B^1}^O$, which implies $(w_1, w_2) \in V_{B^1,B^2}^P \times V_{B^1,B^2}^O$. Since $w_1 \rightsquigarrow w_2$, we have two cases.
  - Case $w_1 \curvearrowright^* w_2$: Then $w_2 \neq f(v_2)$. So the sequence has the next element, i.e. $w_1 \curvearrowright^* w_2 \rightsquigarrow w_3$ for some $w_3$. Since $w_1 \rightsquigarrow w_2 \rightsquigarrow w_3$ and $w_1, w_2, w_3 \in V_{B^1}$, we have $\lambda(w_1) = \lambda(w_3)$, i.e. $w_3 \in V_{B^1}^O$. Hence $v_1 \underset{1}{\rightarrowtail}^* w_3$ and $(v_1, w_3) \in V_{A,B^1}^O \times V_{A,B^1}^P$. Because $s_1$ is a play, we have $v_1 \underset{1}{\rightarrowtail} w_3$. By Corollary 1, we have $v_1 \rightsquigarrow w_3$. Now we have

    $$f(v_1) = v_1 \rightsquigarrow w_3 \rightsquigarrow \ldots \rightsquigarrow w_n \rightsquigarrow w_{n+1} = f(v_2).$$

    Since this sequence is shorter than the original sequence, by the induction hypothesis, we have $v_1 \underset{1}{\rightarrowtail} v_2$.
  - Case $w_1 \curvearrowright^* \overline{w_2}$: Then $w_2 \in V_{B^2}$, which implies $w_2 = f(v_2)$ (i.e. $n = 1$). Hence $w_2 \in V_{B^2}^O$ and $\overline{w_2} \in V_{B^1}^O$. Because $v_1 \rightsquigarrow w_1 \curvearrowright^* \overline{w_2}$ and $s_1$ is a play, we have $v_1 \rightsquigarrow \overline{w_2} = f(v_2)$ as desired.
- Case $v_1 \in V_{B^1}^P$: Then $f(v_1) = \overline{v_1}$ and $\overline{v_1} \rightsquigarrow w_1$. Since $v_1 \in V_{B^1}^P$, we have $\overline{v_1} \in V_{B^2}^P$, and thus $(\overline{v_1}, w_1) \in V_{B^1,B^2}^P \times V_{B^1,B_2}^O$. We have two cases.
  - Case $\overline{v_1} \curvearrowright^* w_1$: Then $w_1 \in V_{B^2}$ and thus $w_1 = f(v_2)$. Since $\overline{v_1} \curvearrowright^* w_1 = f(v_2) = \overline{v_2}$, we have $v_1 \curvearrowright^* v_2$. Since $(\overline{v_1}, \overline{v_2}) \in V_{B^2}^P \times V_{B^2}^O$, we have $(v_1, v_2) \in V_{B^1}^P \times V_{B^2}^O$ and thus $(v_1, v_2) \in V_{A,B^1}^P \times V_{A,B^1}^O$. Since $s_1$ is a play, we have $v_1 \underset{1}{\rightarrowtail} v_2$.
  - Case $\overline{v_1} \curvearrowright^* \overline{w_1}$: Then $w_1 \in V_{B^1}^P$ and thus $w_1 \neq f(v_2)$. So the sequence has the next element, i.e. $\overline{v_1} \rightsquigarrow w_1 \rightsquigarrow w_2$. By the switching condition, $w_2 \in V_{A,B^1}^O$.
    - Case $w_2 \in V_A^P$: Then $w_2 = f(v_2) = v_2$. Since $v_1 \curvearrowright^* w_1 \rightsquigarrow v_2$, $(v_1, v_2) \in V_{A,B^1}^P \times V_{A,B^1}^O$ and $s_1$ is a play, we have $v_1 \underset{1}{\rightarrowtail} v_2$.
    - Case $w_2 \in V_{B^1}^O$: Then the sequence has the next element and thus

      $$v_1 \curvearrowright^* w_1 \rightsquigarrow w_2 \rightsquigarrow w_3 \in V_{B^1,B^2}^O.$$

      - Case $w_2 \curvearrowright^* \overline{w_3}$: Then $w_3 \in V_{B^2}^O$ and thus $w_3 = f(v_2)$, i.e. $v_2 = \overline{w_3}$. Then $\overline{w_3} \in V_{B^1}^O$ and we have $v_1 \curvearrowright^* w_1 \rightsquigarrow w_2 \curvearrowright^* \overline{w_3}$. Since $s_1$ is a play and $(v_1, \overline{w_3}) \in V_{A,B^1}^P \times V_{A,B^1,}^O$, we have $v_1 \underset{1}{\rightarrowtail} \overline{w_3} = v_2$.
      - Case $w_2 \curvearrowright^* w_3$: In this case, we have

        $$\overline{w_3} \rightsquigarrow w_3 \rightsquigarrow w_4 \ldots \rightsquigarrow w_n \rightsquigarrow w_{n+1} = f(v_2).$$

        By the induction hypothesis, we have $w_3 \underset{1}{\rightarrowtail} v_2$. Then $v_1 \curvearrowright^* w_1 \underset{1}{\rightarrowtail} w_2 \curvearrowright^* w_3 \underset{1}{\rightarrowtail} v_2$ and thus $v_1 \underset{1}{\rightarrowtail} v_2$ since $s_1$ is a play.

$\sigma \subseteq \sigma \circ \mathrm{id}_A$ can be proved by the same way. □

**Lemma 9.** *Let $\sigma : A \to B$ be a strategy. Then $\sigma = \mathrm{id}_B \circ \sigma = \sigma \circ \mathrm{id}_A$.*

*Proof.* If $s \in (\mathrm{id}_B \circ \sigma)$, then we have an interaction graph $u \in \mathrm{Int}(A, B^1, B^2)$ such that $u{\restriction}_{A,B^1} \in \sigma$, $u{\restriction}_{B^1,B^2} \in \mathrm{id}_B$ and $u{\restriction}_{A,C} = s$. By Lemma 8, $s = u{\restriction}_{A,B^2} = u{\restriction}_{A,B^1} \in \sigma$.

Suppose $s \in \sigma$. Let $s_B$ be the graph obtained by removing all the causality edge from $s{\upharpoonright_B}$ and let $u' = s \uplus s_B$. The graph $u'$ is the graph obtained by duplicating the $B$ moves (with pointers) in $s$. Assume that $u' = (V, l, \curvearrowright, \rightsquigarrow)$. Suppose that moves from the arena $B$ in $s$ and $s{\upharpoonright_B}$ are relabeled to moves from the arena $B^1$ and arena $B^2$, respectively. For $v \in V_{B^1,B^2}$, $\overline{v}$ denotes the copy of $v$ in the other component. The interaction graph $u$ is defined by $u = (V, l, \curvearrowright, \rightsquigarrow')$, where

$$\rightsquigarrow' := (\rightsquigarrow) \cup \{(v_1, v_2) \in V^P_{B^1,B^2} \times V^O_{B^1,B^2} \mid v_1 \curvearrowright^* v_2 \text{ or } v_1 \curvearrowright^* \overline{v_2}\}.$$

Then we have $u \upharpoonright_{A,B^1} = s \in \sigma$ and $u{\upharpoonright_{B^1,B^2}} = \mathrm{id}_B$. So $u{\upharpoonright_{A,B^2}} \in (\mathrm{id}_B \circ \sigma)$.

The equation $\sigma = \sigma \circ \mathrm{id}_A$ can be proved similarly. □

# C On the Distributive-Closed Freyd Structure (Section 3.4)

## C.1 Monoidal product

Let us redefine the monoidal product of morphisms because the definition written in the body of the paper was informal and vague.

**Definition 23 (Juxtaposition).** *Let $(A, B)$ and $(C, D)$ be arena pairs. Let $s = (V_s, l_s, \curvearrowright_s, \rightsquigarrow_s)$ be a justified graph over $(A, B)$ and $t = (V_t, l_t, \curvearrowright_t, \rightsquigarrow_t)$ be a justified graph over $(C, D)$. Then the* juxtaposition *of $s$ and $t$ is defined by*

$$s \uplus t := (V_s + V_t, l_s + l_t, \curvearrowright_s + \curvearrowright_t, \rightsquigarrow_s + \rightsquigarrow_t).$$

*The juxtaposition of justified graphs over $n$-tuple of arenas (see Appendix B.3) can be defined similarly.*

**Lemma 10.** *Let $s$ and $t$ be plays over an arena pair $(A, B)$. Then $s \uplus t$ is a play over $(A, B)$.*

*Proof.* easy. □

**Definition 24.** *Let $\sigma \subseteq P_{A,B}$ and $\tau \subseteq P_{C,D}$. The monoidal product of $\sigma$ and $\tau$, written $\sigma \odot \tau$ ($\subseteq P_{A \odot C, B \odot D}$), is defined as*

$$\{s \uplus t \mid s \in \sigma, t \in \tau\}.$$

Note that the definition of $\sigma \odot \tau$ is defined using sets of plays that are not necessarily strategies.

**Lemma 11.** *Given strategies $\sigma$ and $\tau$, the monoidal product $\sigma \odot \tau$ is a strategy.*

*Proof.* It suffices to show that $\sigma \odot \tau$ is prefix-closed. Let $r \in \sigma \odot \tau$ and $r' \sqsubseteq r$. By definition, there exists $s, s' \in \sigma$ and $t, t' \in \tau$ such that $r = s \uplus t$ and $r' = s' \uplus t'$. Since $r' \sqsubseteq r$, we have $s' \sqsubseteq s$ and $t' \sqsubseteq t$. It follows that $r' \in \sigma \odot \tau$ by the prefix-closedness of $\sigma$ and $\tau$.

**Theorem 8.** *The operator $\odot$ is a bifunctor from $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$. In other words, $\mathrm{id}_A \odot \mathrm{id}_B = \mathrm{id}_{A \odot B}$ and $(\tau_1 \odot \tau_2) \circ (\sigma_1 \odot \sigma_2) = (\tau_1 \circ \sigma_1) \odot (\tau_2 \circ \sigma_2)$ for all $\sigma_1 \colon A^1 \to B^1$, $\sigma_2 \colon A^2 \to B^2$, $\tau_1 \colon B^1 \to C^1$, and $\tau_2 \colon B^2 \to C^2$.*

*Furhermore, category $\mathcal{P}$ equipped with the tensor product $\odot$ and the unit object $I$ is a well-defined symmetric monoidal category.*

*Proof.* First we show that $\odot$ is a bifunctor. It is clear that $id_A \odot id_B = id_{A \odot B}$.

Now we prove that $\odot$ preserves compositions of morphisms. Let $r$ be a play in $(\tau_1 \odot \tau_2) \circ (\sigma_1 \odot \sigma_2)$ and $u$ be the witness of $r$ i.e. an interaction graph over $(A^1 \odot A^2, B^1 \odot B^2, C^1 \odot C^2)$ such that

1. $u{\restriction_{A^1 \odot A^2, B^1 \odot B^2}} \in \sigma_1 \odot \sigma_2$,
2. $u{\restriction_{B^1 \odot B^2, C^1 \odot C^2}} \in \tau_1 \odot \tau_2$ and
3. $u{\restriction_{A^1 \odot A^2, C^1 \odot C^2}} = r$.

First we show that $u = u_1 \uplus u_2$ for some interaction graphs $u_1 \in \mathrm{Int}(A^1, B^1, C^1)$ and $u_2 \in \mathrm{Int}(A^2, B^2, C^2)$. If $u$ cannot be decomposed as $u_1 \uplus u_2$, it means that there exist $v \in V_{A^1, B^1, C^1}$ and $v' \in V_{A^2, B^2, C^2}$ such that $v \underset{u}{\rightsquigarrow} v'$ or $v' \underset{u}{\rightsquigarrow} v$. Then $u{\restriction_{A^1 \odot A^2, B^1 \odot B^2}}$ or $u{\restriction_{B^1 \odot B^2, C^1 \odot C^2}}$ cannot be decomposed as two disconnect plays $r_1$ and $r_2$ satisfying $r_1 \in \sigma_1$, $r_2 \in \sigma_2$ or $r_1 \in \tau_1$, $r_2 \in \tau_2$, respectively. This contradicts the fact that $u{\restriction_{A^1 \odot A^2, B^1 \odot B^2}} \in \sigma_1 \odot \sigma_2$ and $u{\restriction_{B^1 \odot B^2, C^1 \odot C^2}} \in \tau_1 \odot \tau_2$, so $u = u_1 \uplus u_2$. It is easy to check that $u_i{\restriction_{A_i, B_i}} \in \sigma_i$, $u_i{\restriction_{B_i, C_i}} \in \tau_i$ for all $i \in \{1, 2\}$ and $r = u_1{\restriction_{A^1, C^1}} \uplus u_2{\restriction_{A^2, C^2}}$. Hence, $r \in (\tau_1 \circ \sigma_1) \odot (\tau_2 \circ \sigma_2)$.

The converse inclusion can be easily shown. If $r \in (\tau_1 \circ \sigma_1) \odot (\tau_2 \circ \sigma_2)$ then $r = r_1 \uplus r_2$ such that $r \in \tau_1 \circ \sigma_1$ and $r_2 \in \tau_2 \circ \sigma_2$. Let $u_1$ and $u_2$ be the witnesses of $r_1$ and $r_2$, respectively. Then $u_1 \uplus u_2$ is the witness of $r$ respect to $(\tau_1 \odot \tau_2) \circ (\sigma_1 \odot \sigma_2)$ and thus $r \in (\tau_1 \odot \tau_2) \circ (\sigma_1 \odot \sigma_2)$.

We are left to check that $(\mathcal{P}, \odot, \mathbf{assoc}, \mathbf{unit}, \mathbf{symm})$ is a symmetrical monoidal category, but natural transformations $\mathbf{assoc}_{A,B,C}$, $\mathbf{unit}_A$ and $\mathbf{symm}_{A,B}$ are given by the suitable copycat strategies. Notice that in each case the move set of the domain arena is isomorphic to the move set of the codomain arena. $\square$

### C.2 Category of well-opened strategies

First we formally redefine the strategy $!\sigma$.

**Definition 25.** *Let $s = (V_s, l_s, \underset{s}{\curvearrowright}, \underset{s}{\rightsquigarrow})$ and $t = (V_t, l_t, \underset{t}{\curvearrowright}, \underset{t}{\rightsquigarrow})$ be justified graphs over an arena pair $(A, B)$. The* juxtaposition *of $s$ and $t$ is defined by,*

$$s \uplus' t := (V_s + V_t, [l_s, l_t], \underset{s}{\curvearrowright} + \underset{t}{\curvearrowright}, \underset{s}{\rightsquigarrow} + \underset{t}{\rightsquigarrow}).$$

*Whenever it is clear from context, we abuse notation and write $s \uplus t$ to represent $s \uplus' t$.*

**Definition 26.** *Given a set of plays $\sigma \subseteq P_{A,B}$, we define $!\sigma \subseteq P_{A,B}$ by*

$$!\sigma := \{s_1 \uplus \cdots \uplus s_n \mid n \geq 0, \forall i \leq n.\ s_i \in \sigma\}$$

*where $\uplus$ is the relation defined in Definition 25.*

**Lemma 12.** $!\sigma$ *is a strategy for every well-opened strategy* $\sigma$*.*

*Proof.* Similar to Lemma 11. $\square$

Recall that the composition of well-opened strategies are defined by $\tau \circ_{\mathcal{A}} \sigma := \tau \circ !\sigma$. The fact that the $\tau \circ_{\mathcal{A}} \sigma$ is a well-opened strategy and $\circ_{\mathcal{A}}$ is associative (Lemma 1) is proved by using the following lemma (and the associativity of $\circ$).

**Lemma 13.** *Let* $\sigma\colon A \overset{\bullet}{\to} B$ *and* $\tau\colon B \overset{\bullet}{\to} C$ *be well-opened strategies. Then* $\tau \circ !\sigma$ *is a well-opened strategy and* $!(\tau \circ !\sigma) = !\tau \circ !\sigma$*.*

*Proof.* To show the first claim, assume that there exists a play $s \in \tau \circ !\sigma$ that is not well-open and let $u$ be the witness of $s$. We show the claim using the fact that the initial O-moves (resp. initial P-moves) in $s$ are in $\mathcal{M}_C$ (resp. $\mathcal{M}_A$) because $A$, $B$ and $C$ are all negative arenas. There are three cases to consider.

- Case where $s$ has multiple initial O-moves:
  Since $s$ has multiple initial O-moves, $u{\restriction_{B,C}} \in \tau$ has multiple initial O-moves but this contradicts to the fact that $\tau$ is well-opened.
- Case where there exist a P-node $v_P$ in $s$ such that there is no $v$ satisfying $v \underset{s}{\rightarrow} v_P$ and there is no $v'$ such that $v' \underset{u}{\rightarrow} v_P$:
  Since the label of $v_P$ is a initial P-move, $v_P$ is in $u{\restriction_{A,B}}$. It follows that there is no $v''$ such that $v'' \underset{u}{\rightarrow} {\restriction_{A,B}}\, v_P$ by the assumption that there is no $v'$ such that that $v' \underset{u}{\rightarrow} v_P$ and this contradicts to the fact that $\sigma$ is well-opened.
- Case where there exist a P-node $v_P$ in $s$ such that there is no $v$ satisfying $v \underset{s}{\rightarrow} v_P$ and there exists a node $v'$ such that $v' \underset{u}{\rightarrow} v_P$:
  By assumption the node $v'$ is a node such that $v' \in V_B^O$. Since $v'$ is a P-node in $(A, B)$ then there exists $v_2$ such that $v' \underset{u}{\rightarrow} {\restriction_{A,B}}\, v_2$

To show the second claim we start by showing $!(\tau \circ !\sigma) \subseteq !\tau \circ !\sigma$. Let $r \in !(\tau \circ !\sigma)$ and suppose $r = r_1 \uplus \cdots \uplus r_n$, where $n \geq 0$ and for all $i \in \{1, \ldots, n\}$, $r_i \in \tau \circ !\sigma$. Let $u_i$ be the witness of $r_i$ and $u' = u_1 \uplus \cdots \uplus u_n$. Then we have $u'{\restriction_{A,C}} = r$, $u'{\restriction_{A,B}} = \biguplus_i^n u_i{\restriction_{A,B}} \in !\sigma$ and $u'{\restriction_{B,C}} = \biguplus_i^n u_i{\restriction_{B,C}} \in !\tau$. Therefore $u'$ is a witness of $r$ with respect to $!\tau \circ !\sigma$ and thus $r \in !\tau \circ !\sigma$.

Next we show $!\tau \circ !\sigma \subseteq !(\tau \circ !\sigma)$. Let $r \in !\tau \circ !\sigma$ and $u$ be the witness of $r$. The graph $u$ can be decomposed as $u = u_1 \uplus \cdots \uplus u_n$, where $u_i \in \mathrm{Int}(A, B, C)$ for all $i \in \{1, \ldots, n\}$; otherwise we violate the fact that $u$ is the witness of $r$. Moreover, for each $i \in \{1, \ldots, n\}$, $u_i{\restriction_{A,B}} \in \sigma$ and $u_i{\restriction_{B,C}} \in \tau$. Note that $u_i {\restriction_{A,B}}$ or $u_i{\restriction_{B,C}}$ may be an empty play. Each $u_i$ satisfies $u_i{\restriction_{A,C}} \in \tau \circ \sigma \subseteq \tau \circ !\sigma$. Therefore, $r \in !(\tau \circ !\sigma)$. $\square$

Recall that the *well-opened identity*, i.e. the identity morphism of the category $\mathcal{A}$, is defined by $\mathrm{id}_A^{\mathcal{A}} := \mathrm{id}_A \cap W_{A,A}$. The following lemma shows that $\mathrm{id}_A^{\mathcal{A}}$ is the well-defined identity over an arena $A$.

**Lemma 14.** *If* $\sigma\colon A \overset{\bullet}{\to} B$ *is a well-opened strategy then*

1. $\mathrm{id}_B^{\mathcal{A}} \circ !\sigma = \sigma$ *and*
2. $!\mathrm{id}_A^{\mathcal{A}} = \mathrm{id}_A$*.*

*Therefore,* $\mathrm{id}_B^{\mathcal{A}} \circ_{\mathcal{A}} \sigma = \sigma = \sigma \circ_{\mathcal{A}} \mathrm{id}_A^{\mathcal{A}}$.

*Proof.* First we show the first claim. The proof for $\sigma \subseteq \mathrm{id}_B^{\mathcal{A}} \circ !\sigma$ is similar to that of Lemma 8. Notice that $u\restriction_{B_1,B_2}$, where $u$ is the constructed interaction graph, is well-opened because $\sigma$ is well-opened.

Next we show $\mathrm{id}_B^{\mathcal{A}} \circ !\sigma \subseteq \sigma$. Let $s \in \mathrm{id}_B^{\mathcal{A}} \circ !\sigma$ and $u$ be the witness of $s$. Then by definition we have $u\restriction_{A,B_1} = s_1 \uplus \cdots \uplus s_n$, where $n \geq 0$ and $s_i \in \sigma$ for all $i \in \{1, \dots, n\}$. It suffices to show that $n \leq 1$. Assume that $n \geq 2$. Then there exists more than one initial O-moves in $u\restriction_{A,B_1}$ and because those initial moves are also initial moves in $(B_1, B_2)$ and because copycat graphs copy initial moves, $u\restriction_{B_1,B_2}$ has more than one initial O-moves. This contradicts to the fact that $\mathrm{id}_B^{\mathcal{A}}$ is well-opened, so $n \leq 1$ and $u\restriction_{A,B_2} \in \sigma$.

The second claim is obvious. □

**Theorem 9.** *The category $\mathcal{A}$ whose objects are negative arenas and whose morphisms are well-opened strategy is a well-defined category, with composition and finite products given by $\circ_{\mathcal{A}}$ and $\odot$, respectively.*

*Proof.* The associative axiom and the identity axiom can be shown by Lemma 1 and Lemma 14.

Next we show that $\odot$ is the cartesian product in $\mathcal{A}$. The projections $\pi_1 \colon A \odot B \to A$ and $\pi_2 \colon A \odot B \to B$ are the well-opened copycat strategies over $A$ and $B$, respectively. Given an negative arena $C$ and well-opened strategies $\sigma \colon C \to A$ and $\tau \colon C \to B$, the well-opened strategy defined by $\langle \sigma, \tau \rangle := \{ s \mid s \in \sigma \vee s \in \tau \}$[1] factors $\sigma$ and $\tau$ through $\pi_1$ and $\pi_2$ respectively. □

**Theorem 10.** *The operator* ! *is an identity-on-objects strict symmetrical monoidal functor from $\mathcal{A}$ to $\mathcal{P}$. Therefore, $(\mathcal{A}, \mathcal{P}, !)$ is a Freyd category.*

*Proof.* The fact that ! is a functor is a consequence of 2. of Lemma 13 and 2. of Lemma 14. We are left to check that $!(\sigma \odot \tau) = !\sigma \odot !\tau$, but this is clear by the definition of $\odot$ and !.

### C.3 Closed Freyd category

Here we show that the Freyd category $! \colon \mathcal{A} \to \mathcal{P}$ is closed. We define the functor $A \rightharpoonup (-)$ and prove that it is the right adjoint of $!(-) \odot A$. See Figure 7 (in the body of the paper) for the illustration of the functor $A \rightharpoonup (-)$) and the bijection between $\mathcal{P}(!B \odot A, C)$ and $\mathcal{A}(B, A \rightharpoonup C)$.

**Definition 27.** *Let $A = (\mathcal{M}_A, \lambda_A, \vdash_A)$ and $B = (\mathcal{M}_B, \lambda_B, \vdash_B)$ be arenas. The arena $A \rightharpoonup B := (\mathcal{M}_{A \rightharpoonup B}, \lambda_{A \rightharpoonup B}, \vdash_{A \rightharpoonup B})$ is defined by:*

$$\begin{aligned} \mathcal{M}_{A \rightharpoonup B} &:= \mathcal{M}_A + \mathcal{M}_B + \{*\} \\ \lambda_{A \rightharpoonup B} &:= [\lambda_A^{\perp}, \lambda_B, \{(*, O)\}]. \end{aligned}$$

[1] Strictly speaking, the moves in $s$ needs to be relabelled as in the case of the monoidal product.

*The enabling relation $\vdash_{A \rightharpoonup B}$ is given by: $\star \vdash *$, $* \vdash m$ only if $m \in I_A \cup I_B$ and if $m \vdash_A m'$ or $m \vdash_B m'$ for $m \neq \star$, then $m \vdash_{A \rightharpoonup B} m'$.*

**Definition 28.** *Let $A$, $B$ and $C$ be negative arenas, $s$ be a play over $(B, C)$ and $t$ be a play over $(A_1, A_2)$. Suppose $s \uplus t = (V, l, \curvearrowright, \rightsquigarrow)$. Then we define $A \rightharpoonup (-,-) \colon P_{B,C} \times P_{A_1^\perp, A_2^\perp} \to W_{A \rightharpoonup B, A \rightharpoonup C}$ by*

$$A \rightharpoonup (s,t) := (V \cup \{*_1, *_2\}, l \cup \{(*_1, n), (*_2, m)\}, \curvearrowright \cup \curvearrowright_1 \cup \curvearrowright_2, \rightsquigarrow \cup \rightsquigarrow')$$

*where,*

- $*_1, *_2 \notin V$.
- $n \in I_{A \rightharpoonup B}$ *and* $m \in I_{A \rightharpoonup C}$.
- $\curvearrowright_1 := \{(v, *_1) \mid l(v) \in (I_{A_1} \cup I_B)\}$ *and* $\curvearrowright_2 := \{(v, *_2) \mid l(v) \in (I_{A_2} \cup I_C)\}$.
- $\rightsquigarrow' := \{(*_1, *_2)\} \cup \{(v, *_2) \mid v \in V^P_{B,C} \cup V^P_{A_1^\perp, A_2^\perp}\}$.

*Let $\sigma \colon B \to C$. Then the morphism $A \rightharpoonup \sigma \colon A \rightharpoonup B \overset{\bullet}{\to} A \rightharpoonup C$ is defined as $\{A \rightharpoonup (s,t) \mid s \in \sigma, t \in id_{A^\perp}\}$.*

**Theorem 11.** *Given a negative arena $A$, $A \rightharpoonup -$ is a functor from $\mathcal{P}$ to $\mathcal{A}$.*

**Definition 29.** *Let $A$, $B$ and $C$ be negative arenas and $s = (V, l, \curvearrowright, \rightsquigarrow)$ be a play over $(A \odot B, C)$. Then we define $\mathbf{up} \colon P_{A \odot B, C} \to W_{A, B \rightharpoonup C}$ by*

$$\mathbf{up}(s) := (V \cup \{*\}, l \cup \{(*, m)\}, \curvearrowright \cup \curvearrowright', \rightsquigarrow \cup \rightsquigarrow')$$

*where*

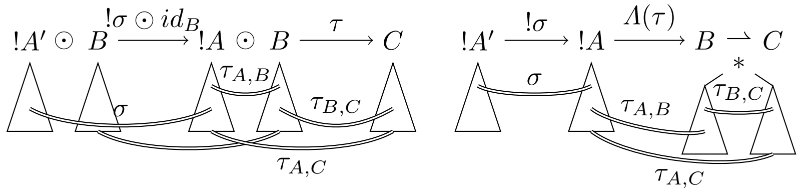
- $* \notin V$.
- $m \in I_{B \rightharpoonup C}$.
- $\curvearrowright' := \{(v, *) \mid l(v) \in I_B \cup I_C\}$.
- $\rightsquigarrow' := \{(v, *) \mid v \in V^P_{A, B \rightharpoonup C}\}$.

*Let $\sigma \colon A \odot B \to C$ be a morphism in $\mathcal{P}$. Then the morphism $\Lambda(\sigma) \colon A \to B \rightharpoonup C$ in $\mathcal{A}$ is defined as $\{\mathbf{up}(s) \mid s \in \sigma\}$.*

Note that **up** is bijective because plays must satisfy the condition (P2). The function $\Lambda$, therefore, is a bijection.

*Remark 4.* Without Condition (P2) $\Lambda$ would not be a bijection and this is the reason for requiring Condition (P2) to plays.

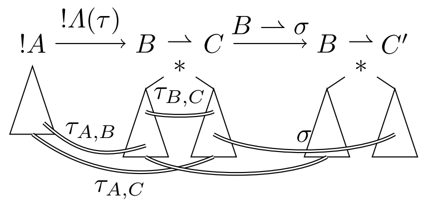
**Theorem 12.** *For all negative arena $B$, $B \rightharpoonup - \colon \mathcal{P} \to \mathcal{A}$ is right adjoint to $!(-) \odot B \colon \mathcal{A} \to \mathcal{P}$ i.e. $\Lambda \colon \mathcal{P}(!A \odot B, C) \cong \mathcal{A}(A, B \rightharpoonup C)$ is an isomorphism natural in $A$ and $C$.*




(a) $\tau \circ (!\sigma \odot \mathrm{id}_B)$

(b) $\Lambda(\tau) \circ !\sigma$

Fig. 11: Supplementary figure for the proof of Theorem 12:
$\Lambda(\tau \circ (!\sigma \odot \mathrm{id}_B)) = \Lambda(\tau) \circ !\sigma$




(a) $\sigma \circ \tau$

(b) $(B \multimap \sigma) \circ !\Lambda(\tau)$

Fig. 12: Supplementary figure for the proof of Theorem 12:
$\Lambda(\sigma \circ \tau) = (B \multimap \sigma) \circ !\Lambda(\tau)$

### C.4 Distributive law

Here the definition of the family of morphisms $\varrho_A$, which was missing in the body of the paper, and the diagram for the axiom $\varrho_A$ must satisfy are given.

**Definition 30.** *Let $A$, $B$ and $C$ be negative arenas. Let $r \in P_{A_1^\perp, A_2^\perp}$, $t \in P_{B_1,B_2}$ and $s \in P_{C_1,C_2}$ and suppose $r \uplus s \uplus t = (V, l, \curvearrowright, \rightsquigarrow)$. We define* $\mathbf{dist}\colon P_{A_1^\perp, A_2^\perp} \times P_{B_1,B_2} \times P_{C_1,C_2} \to P_{!(A \multimap (B \odot C)), B \odot !(A \multimap C)}$ *by*

$$\mathbf{dist}(r,s,t) := (V \cup \{*_1, *_2\}, l \cup \{(*_1, m_1), (*_2, m_2)\}, \curvearrowright \cup \curvearrowright_1 \cup \curvearrowright_2, \rightsquigarrow \cup \rightsquigarrow' \cup (*_1, *_2))$$

*where*

- $*_1, *_2 \notin V$.
- $m_1 \in I_{!(A \multimap (B \odot C))}$ *and* $m_2 \in I_{!(A \multimap C)}$.
- $\curvearrowright_1 := \{(v, *_1) \mid l(v) \in I_{A_1} \cup I_{B_1} \cup I_{C_1}\}$ *and* $\curvearrowright_2 := \{(v, *_2) \mid l(v) \in I_{A_2} \cup I_{C_2}\}$.
- $\rightsquigarrow' := \{(v, *_2) \mid v \in V^P_{A_1^\perp, A_2^\perp} \cup V^P_{B_1,B_2} \cup V^P_{C_1,C_2}\}$.

*The morphism* $\varrho_{A,B,C}\colon !(A \multimap (B \odot C)) \to B \odot !(A \multimap C)$ *in* $\mathcal{P}$ *is defined as* $\{\mathbf{dist}(r,s,t) \mid r \in \mathrm{id}_{A^\perp}, s \in \mathrm{id}_B, t \in \mathrm{id}_C\}$.

Given a strategy $\sigma$, informally, the strategy $\varrho_{A,B,C} \circ \sigma$ is a set of plays obtained by relabeling initial moves and removing the justification pointers from the initial moves in $B$ for each play in $\sigma$.

**Theorem 13.** *The family of morphisms $\varrho_{A,B,C}$ is a natural transformation from $!(A \rightharpoonup (- \odot -)) \colon \mathcal{A} \to \mathcal{P}$ to $- \odot !(A \rightharpoonup -) \colon \mathcal{A} \to \mathcal{P}$ for each object $A$.*

*The natural transformation $\varrho_{A,B,C}$ satisfies the following diagrams:*

$$
\begin{tikzcd}
!(A_1 \rightharpoonup (B_1 \odot C_1)) \odot A_2 \arrow[rr, "\mathbf{app}_{A,B\odot C}"] \arrow[d, "\varrho_{A,B,C} \odot id_A"] & & B_4 \odot C_4 \\
(B_2 \odot !(A_3 \rightharpoonup C_2)) \odot A_4 \arrow[r, "\mathbf{assoc}_{B,!(A \rightharpoonup C),A}"] & B_3 \odot (!(A_5 \rightharpoonup C_3) \odot A_6) \arrow[ur, "id_B \odot \mathbf{app}_{A,C}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
!(A_1 \rightharpoonup ((B_1 \odot C_1) \odot D_1)) \arrow[r, "!(A \rightharpoonup \mathbf{assoc}_{B,C,D})"] \arrow[d, "\varrho_{A,B\odot C,D}"] & !(A_5 \rightharpoonup (B_5 \odot (C_5 \odot D_5))) \arrow[r, "\varrho_{A,B,C\odot D}"] & B_4 \odot !(A_4 \rightharpoonup (C_4 \odot D_4)) \arrow[d, "id_B \odot \varrho_{A,C,D}"] \\
(B_2 \odot C_2) \odot !(A_2 \rightharpoonup D_2) \arrow[rr, "\mathbf{assoc}_{B,C,!(A \rightharpoonup D)}"] & & B_3 \odot (C_3 \odot !(A_3 \rightharpoonup D_3))
\end{tikzcd}
$$

*Here the subscripts are used to distinguish the arenas in different positions. Therefore, the Freyd category $! \colon \mathcal{A} \to \mathcal{P}$ is distributive-closed.*

# D Supplementary Materials for Section 3.5

## D.1 Laird's model

Here we briefly review the definition of the game model of Laird [24]. We call plays (resp. strategies) in [24] interleaving plays (resp. interleaving strategies) in order to distinguish these notions from ours. Let $(A, B)$ be an arena pair. An *interleaving play* over $(A, B)$ is a sequence of moves equipped with justification pointers. Formally it is a triple $\hat{s} = (\#\hat{s}, l_{\hat{s}}, \rho_{\hat{s}})$ of the length $\#\hat{s}$ of $\hat{s}$, a function $l_{\hat{s}} : \{1, \dots, \#\hat{s}\} \to \mathcal{M}_{A,B}$ and a partial function $\rho_{\hat{s}} \colon \{1, \dots, \#\hat{s}\} \to \{1, \dots, \#\hat{s}\}$ subject to the following conditions: (1) $\rho_{\hat{s}}(i) < i$ if $\rho_{\hat{s}}(i)$ is defined, (2) the pointer respects the enabling relation, i.e. $l_{\hat{s}}(\rho_{\hat{s}}(i)) \vdash_{A,B} l_{\hat{s}}(i)$ if $\rho_{\hat{s}}(i)$ is defined, and (3) $\rho_{\hat{s}}(i)$ is undefined only if $l_{\hat{s}}(i)$ is an initial move. As usual, we often write an interleaving play $\hat{s}$ as a sequence $\hat{s} = m_1 m_2 \dots m_k$ of moves, leaving the justification pointers implicit. A set $\hat{\sigma}$ of interleaving plays is an *interleaving strategy* if it is non-empty and satisfies the following conditions:

**(L1)** If $\hat{s}'$ is a prefix of $\hat{s} \in \hat{\sigma}$, then $\hat{s}' \in \hat{\sigma}$.
**(L2)** If $m$ is an O-move and $\hat{s}_1 \hat{s}_2 \in \hat{\sigma}$, then $\hat{s}_1 m \hat{s}_2 \in \hat{\sigma}$.
**(L3)** If $m$ is a P-move and $\hat{s}_1 m m' \hat{s}_2 \in \hat{\sigma}$, then $\hat{s}_1 m' m \hat{s}_2 \in \hat{\sigma}$.
**(L4)** If $m$ is an O-move and $\hat{s}_1 m' m \hat{s}_2 \in \hat{\sigma}$, then $\hat{s}_1 m m' \hat{s}_2 \in \hat{\sigma}$.

As described in [24], negative arenas and interleaving strategies can be organised into a distributive-closed Freyd category, which we write as $\mathcal{P}_{\mathrm{L}}$.

### D.2 Sequentialising DAG-based plays

From a given (concurrent) play $s = (V_s, l_s, \underset{s}{\curvearrowright}, \underset{s}{\rightarrowtail})$, an interleaving play is obtained by lining up nodes in $V_s$ in such a way that if $v_1 \underset{s}{\to} v_2$, then $v_2$ appears before $v_1$. A *linearisation function* is a bijection $f \colon V_s \xrightarrow{\cong} \{1, \dots, \#V_s\}$ (where $\#V_s$ is the number of nodes in $V_s$) such that $v_1 \to v_2$ implies $f(v_1) > f(v_2)$. Given a linearisation function $f$, we obtain an interleaving play $\hat{s}_f := (\#V_s, l \circ f^{-1}, f(\underset{s}{\curvearrowright}))$, where $f(\underset{s}{\curvearrowright})$ is the partial function whose graph is $\{(f(v_1), f(v_2)) \mid v_1, v_2 \in V_s,\ v_1 \underset{s}{\curvearrowright} v_2\}$. We define

$$|s| := \{\hat{s}_f \mid f \text{ is a linearisation function of } g\}.$$

This operation is extended to sets of (concurrent) plays $\sigma$ by

$$|\sigma| := \mathsf{cl}(\bigcup \{|s| \mid s \in \sigma\}),$$

where $\mathsf{cl}(\hat{\sigma})$ is the closure of $\hat{\sigma}$ by Condition (L2).[2]

**Lemma 15.** *Let $s$ and $\sigma$ be a (concurrent) play and a set of (concurrent) plays over $(A, B)$, respectively. Then*

1. *$|s|$ satisfies the conditions (L3) and (L4), and*
2. *If $\sigma$ is a strategy of $\mathcal{P}$, then $|\sigma|$ is a strategy of $\mathcal{P}_{\mathrm{L}}$.*

### D.3 Proof of Theorem 3

Given a (concurrent) strategy $\sigma$, we write $|\sigma|_0$ for the set of sequential plays defined by

$$|\sigma|_0 := \bigcup \{|s| \mid s \in \sigma\}.$$

So $|\sigma| = \mathsf{cl}(|\sigma|_0)$. For every concurrent strategy $\sigma$, $|\sigma|_0$ satisfies (L1), (L3) and (L4) (but not necessarily (L2)).

**Lemma 16.** *Let $\hat{\sigma} : A \to B$ and $\hat{\tau} : B \to C$ be sets of sequential plays that satisfies (L1), (L3) and (L4) (but not necessarily (L2)). Then $\mathsf{cl}(\hat{\tau} \circ \hat{\sigma}) = \mathsf{cl}(\hat{\tau}) \circ \mathsf{cl}(\hat{\sigma})$.*

*Proof.* Since $\hat{\sigma} \subseteq \mathsf{cl}(\hat{\sigma})$, we have

$$\hat{\tau} \circ \hat{\sigma} \subseteq \mathsf{cl}(\hat{\tau}) \circ \mathsf{cl}(\hat{\sigma}).$$

Since $\mathsf{cl}$ is monotone, $\mathsf{cl}(\hat{\sigma})$ and $\mathsf{cl}(\hat{\tau})$ are strategies and the composition strategies is a strategy [24], we have

$$\mathsf{cl}(\hat{\tau} \circ \hat{\sigma}) \subseteq \mathsf{cl}(\mathsf{cl}(\hat{\tau}) \circ \mathsf{cl}(\hat{\sigma})) = \mathsf{cl}(\hat{\tau}) \circ \mathsf{cl}(\hat{\sigma}).$$

[2] The operator $\mathsf{cl}$ is needed to fill an inessential gap between two representations of strategies: strategies as sets of even-length plays and those as sets of plays that satisfies the *contingent completeness* [21]. We use the former but Laird used the latter.

Assume that $\hat{s} \in \mathsf{cl}(\hat{\tau}) \circ \mathsf{cl}(\hat{\sigma})$. Then we have an interaction sequence $\hat{u}$ such that $\hat{u}\restriction_{A,B} \in \mathsf{cl}(\hat{\sigma})$, $\hat{u}\restriction_{B,C} \in \mathsf{cl}(\hat{\tau})$ and $\hat{u}\restriction_{A,C} = \hat{s}$. Since $\hat{u}\restriction_{A,B} \in \mathsf{cl}(\hat{\sigma})$, we have $\hat{t}_1 \in \hat{\sigma}$ obtained by removing some O-moves from $\hat{u}\restriction_{A,B}$. Similarly we have $\hat{t}_2 \in \hat{\tau}$ obtained by removing some O-moves from $\hat{u}\restriction_{A,B}$. Let $\hat{u}_0$ be the subsequence of $\hat{u}$ consisting of

- $A$-moves in $\hat{t}_1$,
- $B$-moves in both $\hat{t}_1$ and $\hat{t}_2$, and
- $C$-moves in $\hat{t}_2$.

In other words, $\hat{u}_0$ is obtained by removing moves that is removed from $\hat{u}\restriction_{A,B}$ to obtain $\hat{t}_1$ or from $\hat{u}\restriction_{B,C}$ to obtain $\hat{t}_2$. Then $\hat{u}_0\restriction_{A,B}$ is obtained by removing some P-moves from $\hat{t}_1$ and $\hat{u}_0\restriction_{B,C}$ is obtained by removing some P-moves from $\hat{t}_2$. Hence, by (L1) and (L3), we have $\hat{u}_0\restriction_{A,B} \in \hat{\sigma}$ and $\hat{u}_0\restriction_{B,C} \in \hat{\tau}$. So $\hat{u}_0\restriction_{A,C} \in \hat{\tau} \circ \hat{\sigma}$. By construction, $\hat{u}_0\restriction_{A,C}$ is obtained by removing O-moves from $\hat{u}\restriction_{A,C}$, and thus $\hat{u}\restriction_{A,C} \in \mathsf{cl}(\hat{\tau} \circ \hat{\sigma})$. □

**Lemma 17.** *Let $\sigma : A \to B$ and $\tau : B \to C$ be (concurrent) strategies. Then $|\tau|_0 \circ |\sigma|_0 = |\tau \circ \sigma|_0$.*

*Proof.* Assume that $\hat{s} \in |\tau \circ \sigma|_0$. Then we have $s \in \tau \circ \sigma$ such that $\hat{s} \in |s|$. Let $u \in \mathrm{Int}(A, B, C)$ be the interaction graph such that $u\restriction_{A,B} \in \sigma$, $u\restriction_{B,C} \in \tau$ and $u\restriction_{A,C} = s$. Let $V$ be the set of nodes of $u$. The sequential play $\hat{s}$ introduces the linear order on $V_{A,C}$, which we write as $\preceq_0$. This linear order $\preceq_0$ is compatible with $\underset{u}{\rightarrow}^*$ in the sense that $v \underset{u}{\rightarrow}^* v'$ implies $v \succeq_0 v'$ for every $v, v' \in V_{A,C}$. Since $\underset{u}{\rightarrow}^*$ is a partial order, one can extend $\preceq_0$ to a linear order on $V$. This linear order determines an interaction sequence $\hat{u}$ such that $\hat{u}\restriction_{A,B} \in |\sigma|_0$ and $\hat{u}\restriction_{B,C} \in |\tau|_0$. Hence $\hat{u}\restriction_{A,C} \in |\tau|_0 \circ |\sigma|_0$. Obviously $\hat{u}\restriction_{A,C} = \hat{s}$.

Assume that $\hat{s} \in |\tau|_0 \circ |\sigma|_0$. Then we have an interaction sequence $\hat{u}$ such that $\hat{u}\restriction_{A,B} \in |\sigma|_0$, $\hat{u}\restriction_{B,C} \in |\tau|_0$ and $\hat{u}\restriction_{A,C} = \hat{s}$. So we have $s_1 \in \sigma$ and $s_2 \in \tau$ such that $\hat{u}\restriction_{A,B} \in |s_1|$ and $\hat{u}\restriction_{B,C} \in |s_2|$. We can assume without loss of generality that nodes in $s_1$ and $s_2$ are natural numbers indicating positions in $\hat{u}$. Then $i \underset{s_1}{\rightarrow} j$ or $i \underset{s_2}{\rightarrow} j$ implies $i > j$. We define an interaction graph $u = (V, l, \curvearrowright, \rightsquigarrow)$ as follows.

- $V = \{1, 2, \dots, n\}$ where $n$ is the length of $\hat{u}$.
- $l(i)$ is the name of the move at the $i$-th position in $\hat{u}$.
- $i \curvearrowright j$ if and only if $i$-th move points to $j$-th move in $\hat{u}$. Equivalently $i \curvearrowright j$ if and only if $i \underset{s_1}{\curvearrowright} j$ or $i \underset{s_2}{\curvearrowright} j$.
- $i \rightsquigarrow j$ if and only if $i \underset{s_1}{\rightsquigarrow} j$ or $i \underset{s_2}{\rightsquigarrow} j$.

Then $u$ is indeed an interaction graph. (Acyclicity of $u$ comes from the fact that $i \underset{u}{\rightarrow} j$ implies $i > j$.) It is easy to see that $u\restriction_{A,B} = s_1$ and $u\restriction_{B,C} = s_2$. Hence $u\restriction_{A,C} \in \tau \circ \sigma$. Since $\hat{s} \in |u\restriction_{A,C}|$, we have $\hat{s} \in \mathsf{cl}(\tau \circ \sigma)$. □

**Proof of Theorem 3** $|-|$ is a functor because of Lemmas 16 and 17. Obviously $|\bot| = \bot$. If $\sigma \neq \bot$, then there exists a play $s \in \sigma$ which has a P-move because of (P3). Since $|s|$ is nonempty and $\hat{s} \in |s|$ contains a P-move, we have $|\sigma| \neq \bot$. Preservation of other structures are lengthy but routine.

## E Supplementary Materials for Section 4

### E.1 Complete definition of the interpretation

The complete definition of the interpretation of processes, which we have omitted in the body of the paper, is given in Figure 13.

$$
\begin{aligned}
[\![\Gamma \vdash P\{z/x, z/y\}; \Sigma, z : T]\!] &= (\mathrm{id}_{[\![\Sigma]\!]} \odot !\nabla_{[\![T]\!]}) \circ [\![P]\!] \\
[\![\Gamma, \bar{z} : T \vdash P\{\bar{z}/\bar{x}, \bar{z}/\bar{y}\}; \Sigma]\!] &= [\![P]\!] \circ (\mathrm{id}_{[\![\Gamma]\!]} \odot !\Delta_{[\![T]\!]}) \\
[\![\Gamma \vdash P; \Sigma, y : T, x : S, \Sigma']\!] &= (\mathrm{id}_{[\![\Sigma]\!]} \odot \mathbf{symm}_{[\![S]\!],[\![T]\!]} \odot \mathrm{id}_{[\![\Sigma']\!]}) \circ [\![P]\!] \\
[\![\Gamma, \bar{y} : T, \bar{x} : S, \Gamma' \vdash P; \Sigma]\!] &= [\![P]\!] \circ (\mathrm{id}_{[\![\Gamma]\!]} \odot \mathbf{symm}_{[\![T]\!],[\![S]\!]} \odot \mathrm{id}_{[\![\Gamma']\!]}) \\
[\![\Gamma \vdash \mathbf{0}; \Sigma]\!] &= \bot_{[\![\Gamma]\!],[\![\Sigma]\!]} \\
[\![\Gamma, \Gamma' \vdash P|Q; \Sigma, \Sigma']\!] &= [\![P]\!] \odot [\![Q]\!] \\
[\![\Gamma, \bar{x} : \mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}], \bar{\boldsymbol{y}} : \boldsymbol{S} \vdash \bar{x}\langle \bar{\boldsymbol{y}}, \boldsymbol{z}\rangle; \Sigma, \boldsymbol{z} : \boldsymbol{T}]\!] &= \bot_{[\![\Gamma]\!],[\![\Sigma]\!]} \odot \mathbf{app}_{[\![\boldsymbol{S}]\!],[\![\boldsymbol{T}]\!]} \\
[\![\Gamma \vdash x(\bar{\boldsymbol{y}}, \bar{z}).P; \Sigma, x : \mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}]]\!] &= (\mathrm{id}_{[\![\Sigma]\!]} \odot der_{[\![(\boldsymbol{S},\boldsymbol{T})]\!]}) \circ [\![!x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P]\!] \\
[\![\Gamma \vdash !x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P; \Sigma, x : \mathbf{ch}[\boldsymbol{S}, \boldsymbol{T}]]\!] &= \varrho_{[\![\boldsymbol{S}]\!],[\![\Sigma]\!],[\![\boldsymbol{T}]\!]} \circ !\Lambda([\![P]\!]) \\
[\![\Gamma \vdash \nu(\bar{x}, y).P; \Sigma]\!] &= Tr^{[\![T]\!]}_{[\![\Gamma]\!],[\![\Sigma]\!]}([\![P]\!])
\end{aligned}
$$

Fig. 13: Interpretation of processes. (Complete version.)

**Additional properties to prove Theorem 5**

In order to prove Theorem 5 Laird [24] has used some additional properties except for the properties we can obtain from the distributive-closed Freyd category. Here we write them out and check that our model satisfies the desiring properties. (See [24] or Appendix E.2 for the actual proof.)

*Remark 5.* It does not mean that the following properties are the minimal requirements to prove Theorem 5. Requiring the following properties is sufficient to prove Theorem 5, however.

First the following order (or equality) between morphisms are used.

- For all $\sigma$, $\bot \subseteq \sigma$.
- $der_A \subseteq \mathrm{id}_A$.
- For all $i \in \{1, 2\}$, $\pi_i \subseteq \nabla$, where $\pi_i$ is the projection of the product.
- $\mathrm{id}_A \subseteq !\Delta_A \circ !\nabla_A$ and $\mathrm{id}_A = !\nabla_A \circ !\Delta_A$.

It is easy to check that our model also satisfies these properties.

To prove (2) of Theorem 5 we also use the following equation:

$$\mathbf{app}_{A,B} \circ (der_A \odot \mathrm{id}_B) = \mathbf{app}_{A,B}.$$

In our model (and in the interleaving model) this equation is shown using the fact that *der* is the well-opened identity and **app** is the "almost copycat" strategy.

Finally to show that $[\![!P]\!] = [\![P|!P]\!]$ we use the following equations:

$$(\mathrm{id}_A \odot \mathbf{symm}_{B,A} \odot \mathrm{id}_B); (!\nabla_A \odot !\nabla_B) = !\nabla_{A \odot B}$$
$$(!\Delta_A \odot !\Delta_B); (id_A \odot \mathbf{symm}_{A,B} \odot id_B) = !\Delta_{A \odot B}$$

This is shown by the fact that id, **symm**, $\Delta$ and $\nabla$ are defined using copycat strategies.

### E.2 Proof sketch of Theorem 5

In this section we sketch the proof for the (weak) soundness property with respect to the reduction. The proofs are by rewriting the equations using the categorical properties and the additional properties we mentioned in the previous section. Especially the axioms for the trace operator is heavily used.

The following lemmas are used to prove the soundness.

*Remark 6.* In this section we use the diagrammatic order for composition; we use ; and $;_{\mathcal{A}}$ for the composition in $\mathcal{P}$ and $\mathcal{A}$ respectively.

**Lemma 18.** $(\mathrm{id}_A \odot \mathbf{symm}_{B,A} \odot \mathrm{id}_B); (!\nabla_A \odot !\nabla_B) = !\nabla_{A \odot B}$ *and* $(!\Delta_A \odot !\Delta_B); (\mathrm{id}_A \odot \mathbf{symm}_{A,B} \odot \mathrm{id}_B) = !\Delta_{A,B}$.

**Lemma 19.** *Given* $f \in \mathcal{A}(A, B \odot C)$*, we have* $!f = !\Delta_A; ((!f; (\mathrm{id}_B \odot der_C)) \odot !f); !\nabla_{B \odot C}$.

*Proof.*

$$\begin{aligned}
!f &= !(\langle \bot_{A,B\odot C}, f\rangle ;_{\mathcal{A}} \pi_2) \\
&= !(\Delta_A ;_{\mathcal{A}} (\bot_{A,B\odot C} \odot f) ;_{\mathcal{A}} \pi_2) \\
&= !\Delta_A; (\bot_{A,B\odot C} \odot !f); !\pi_2 \\
&\subseteq !\Delta_A; ((!f; (\mathrm{id}_B \odot der_C)) \odot !f); !\pi_2 && (\forall f.\ \bot \subseteq f) \\
&\subseteq !\Delta_A; ((!f; (\mathrm{id}_B \odot der_C)) \odot !f); !\nabla_{B\odot C} && (\pi_2 \subseteq \pi_1 \cup \pi_2 = \nabla) \\
&\subseteq !\Delta_A; (!f \odot !f); !\nabla_{B\odot C} && (der \subseteq \mathrm{id}) \\
&= !(\Delta_A ;_{\mathcal{A}} (f \odot f) ;_{\mathcal{A}} \nabla_{B\odot C}) \\
&= !(\langle f, f\rangle ;_{\mathcal{A}} (\pi_1 \cup \pi_2)) \\
&= !(f \cup f) \\
&= !f
\end{aligned}$$

□

First we show the soundness with respect to the structural congruence.

**Lemma 20.** *If $P \equiv Q$ then $[\![\Gamma \vdash P; \Sigma]\!] = [\![\Gamma \vdash Q; \Sigma]\!]$ (modulo* **assoc** *and* **unit***).*

*Proof.* (Sketch.)

– Case where $P|Q \equiv Q|P$:

$$
\begin{aligned}
&[\![\Gamma, \Gamma' \vdash Q|P; \Sigma, \Sigma']\!] \\
&= \mathbf{symm}_{\Gamma,\Gamma'}; [\![\Gamma', \Gamma \vdash Q|P; \Sigma', \Sigma]\!]; \mathbf{symm}_{\Sigma',\Sigma} \\
&= \mathbf{symm}_{\Gamma,\Gamma'}; ([\![\Gamma' \vdash Q; \Sigma']\!] \odot [\![\Gamma \vdash P; \Sigma]\!]); \mathbf{symm}_{\Sigma',\Sigma} \\
&= ([\![\Gamma \vdash P; \Sigma]\!] \odot [\![\Gamma' \vdash Q; \Sigma']\!]); \mathbf{symm}_{\Sigma,\Sigma'}; \mathbf{symm}_{\Sigma',\Sigma} \\
&\hfill (\text{naturality of } \mathbf{symm}) \\
&= ([\![\Gamma \vdash P; \Sigma]\!] \odot [\![\Gamma' \vdash Q; \Sigma']\!]) \qquad (\mathbf{symm}; \mathbf{symm} = \mathrm{id}) \\
&= [\![\Gamma, \Gamma' \vdash P|Q; \Sigma, \Sigma']\!].
\end{aligned}
$$

– Case where $\mathbf{0}|P \equiv P$:
trivial.

– Case where $P|(Q|R) \equiv (P|Q)|R$:
trivial.

– Case where $\nu x.\nu y.P \equiv \nu y.\nu x.Q$:

$$
\begin{aligned}
&[\![\Gamma, \nu x.\nu y.P; \Sigma]\!] \\
&= Tr^{T}_{\Gamma,\Sigma}([\![\Gamma, \bar{x} : T \vdash \nu y.P; \Sigma, x : T]\!]) \\
&= Tr^{T}_{\Gamma,\Sigma}(Tr^{S}_{\Gamma\odot T,\Sigma\odot T}([\![\Gamma, \bar{x} : T, \bar{y} : S \vdash P; \Sigma, x : T, y : S]\!])) \\
&= Tr^{T\odot S}_{\Gamma,\Sigma}([\![\Gamma, \bar{x} : T, \bar{y} : S \vdash P; \Sigma, x : T, y : S]\!]) \qquad (\text{vanishing}) \\
&= Tr^{T\odot S}_{\Gamma,\Sigma}((\mathrm{id}_\Gamma \odot \mathbf{symm}_{T,S}); [\![\Gamma, \bar{y} : S, \bar{x} : T \vdash P; \Sigma, y : S, x : T]\!](\mathrm{id}_\Sigma \odot \mathbf{symm}_{S,T})) \\
&= Tr^{S\odot T}_{\Gamma,\Sigma}((\mathrm{id}_\Gamma \odot \mathbf{symm}_{S,T}); (\mathrm{id}_\Gamma \odot \mathbf{symm}_{T,S}); [\![\Gamma, \bar{y} : S, \bar{x} : T \vdash P; \Sigma, y : S, x : T]\!]) \\
&= Tr^{S\odot T}_{\Gamma,\Sigma}([\![\Gamma, \bar{y} : S, \bar{x} : T \vdash P; \Sigma, y : S, x : T]\!]) \qquad (\mathbf{symm}; \mathbf{symm} = \mathrm{id}) \\
&= Tr^{S}_{\Gamma,\Sigma}(Tr^{T}_{\Gamma\odot S,\Sigma\odot S}([\![\Gamma, \bar{y} : S, \bar{x} : T \vdash P; \Sigma, y : S, x : T]\!])) \qquad (\text{vanishing}) \\
&= [\![\Gamma, \nu y.\nu x.P; \Sigma]\!]
\end{aligned}
$$

.

– Case where $(\nu x.P)|Q \equiv \nu x.(P|Q)$:

$$
\begin{aligned}
&[\![\Gamma, \Gamma' \vdash \nu x.(P|Q); \Sigma, \Sigma']\!] \\
&= Tr^{T}_{\Gamma\odot\Gamma',\Sigma\odot\Sigma'}([\![\Gamma, \Gamma', \bar{x} : T \vdash P|Q; \Sigma, \Sigma', x : T]\!]) \\
&= Tr^{T}_{\Gamma\odot\Gamma',\Sigma\odot\Sigma'}; ((\mathrm{id}_\Gamma \odot \mathbf{symm}_{\Gamma',T})[\![\Gamma, \bar{x} : T, \Gamma' \vdash P|Q; \Sigma, x : T, \Sigma']\!]; \\
&\qquad (\mathrm{id}_\Sigma \odot \mathbf{symm}_{T,\Sigma'})) \\
&= Tr^{T}_{\Gamma\odot\Gamma',\Sigma\odot\Sigma'}((\mathrm{id}_\Gamma \odot \mathbf{symm}_{\Gamma',T}); ([\![\Gamma, \bar{x} : T \vdash P; \Sigma, x : T]\!] \odot [\![\Gamma' \vdash Q; \Sigma']\!]); \\
&\qquad (\mathrm{id}_\Sigma \odot \mathbf{symm}_{T,\Sigma'})) \\
&= Tr^{T}_{\Gamma,\Sigma}([\![\Gamma, \bar{x} : T \vdash P; \Sigma, x : T]\!]) \odot [\![\Gamma' \vdash Q; \Sigma']\!] \\
&\hfill ((\text{generalized}) \text{ superposing}) \\
&= [\![\Gamma, \Gamma' \vdash (\nu x.P)|Q; \Sigma, \Sigma']\!]
\end{aligned}
$$

– Case where $!P \equiv P|!P$:

$$
\begin{aligned}
&[\![\Gamma \vdash x(\boldsymbol{y},\boldsymbol{z}).P|!x(\boldsymbol{y},\boldsymbol{z}).P;\Sigma, x:T]\!]\\
&= !\Delta_\Gamma; [\![\Gamma,\Gamma \vdash x(\boldsymbol{y},\boldsymbol{z}).P|!x(\boldsymbol{y},\boldsymbol{z}).P;\Sigma, x:T,\Sigma, x:T]\!]; !\nabla_{\Sigma\odot T}\\
&\hfill \text{(Lemma 18)}\\
&= !\Delta_\Gamma; ([\![\Gamma \vdash x(\boldsymbol{y},\boldsymbol{z}).P;\Sigma, x:T]\!] \odot [\![\Gamma \vdash !x(\boldsymbol{y},\boldsymbol{z}).P;\Sigma, x:T]\!]); !\nabla_{\Sigma\odot T}\\
&= !\Delta_\Gamma; (([\![\Gamma \vdash !x(\boldsymbol{y},\boldsymbol{z}).P;\Sigma, x:T]\!]; (id_\Sigma \odot der)) \odot [\![\Gamma \vdash !x(\boldsymbol{y},\boldsymbol{z}).P;\Sigma, x:T]\!]); !\nabla_{\Sigma\odot T}\\
&\hfill \text{(definition of } [\![!P]\!])\\
&= [\![\Gamma \vdash !x(\boldsymbol{y},\boldsymbol{z}).P;\Sigma, x:T]\!] \qquad \text{(Lemma 19)}
\end{aligned}
$$

.

□

**Lemma 21.** *If $P \longrightarrow Q$ then $[\![P]\!] \supseteq [\![Q]\!]$.*

*Proof.* (Sketch.) In order to show the soundness we first show the following: $[\![\Gamma \vdash \nu x.\nu y.P;\Sigma]\!] \subseteq [\![\Gamma \vdash \nu x.P[x/y,\bar{x}/\bar{y}];\Sigma]\!]$, where $x:T$ and $y:T$:

$$
\begin{aligned}
&[\![\Gamma \vdash \nu x.\nu y.P\Sigma]\!]\\
&= Tr^T_{\Gamma,\Sigma}(Tr^T_{\Gamma\odot T,\Sigma\odot T}([\![\Gamma,\bar{x}:T,\bar{y}:T \vdash P;\Sigma,x:T,y:T]\!]))\\
&= Tr^{T\odot T}_{\Sigma,\Gamma}([\![\Gamma,\bar{x}:T,\bar{y}:T \vdash P;\Sigma,x:T,y:T]\!]) \qquad \text{(vanishing)}\\
&= Tr^{T\odot T}_{\Sigma,\Gamma}((\mathrm{id}_\Sigma \odot \mathrm{id}_{B\odot B}); [\![\Gamma,\bar{x}:T,\bar{y}:T \vdash P;\Sigma,x:T,y:T]\!])\\
&\subseteq Tr^{T\odot T}_{\Sigma,\Gamma}((id_\Sigma \odot (!\nabla; !\Delta)); [\![\Gamma,\bar{x}:T,\bar{y}:T \vdash P;\Sigma,x:T,y:T]\!]) \qquad (\mathrm{id} \subseteq !\nabla; !\Delta)\\
&= Tr^{T}_{\Sigma,\Gamma}((\mathrm{id}_\Gamma \odot !\nabla_T); [\![\Gamma,\bar{x}:T,\bar{y}:T \vdash P;\Sigma,x:T,y:T]\!]; (\mathrm{id}_\Sigma \odot !\Delta_T))\\
&\hfill \text{(sliding)}\\
&= Tr^{T}_{\Sigma,\Gamma}([\![\Gamma,\bar{x}:T \vdash P[x/y,\bar{x}/\bar{y}];\Sigma,x:T]\!])\\
&= [\![\Gamma \vdash \nu x.P[x/y,\bar{x}/\bar{y}];\Sigma]\!]
\end{aligned}
$$

Next we show that the following equation holds

$$
\begin{aligned}
([\![P']\!] :=\ )&[\![\Gamma,\bar{\boldsymbol{a}}:S \vdash \nu c.(\bar{c}\langle\bar{\boldsymbol{a}},\boldsymbol{b}\rangle|c(\bar{\boldsymbol{y}},\boldsymbol{z}).P);\Sigma,\boldsymbol{b}:T]\!]\\
&= [\![\Gamma,\bar{\boldsymbol{a}}:S \vdash P[\boldsymbol{b}/\boldsymbol{z},\bar{\boldsymbol{a}}/\bar{\boldsymbol{y}}];\Sigma,\boldsymbol{b}:T]\!]
\end{aligned}
$$

Here we assume that $c \notin \mathbf{fn}(P)$.

$$
\begin{aligned}
[\![P']\!] &= Tr^{S\rightharpoonup T}_{\Gamma\odot S,\Sigma\odot T}([\![\Gamma,\bar{\boldsymbol{a}}:S,\bar{c}:S\rightharpoonup T \vdash \bar{c}\langle\bar{\boldsymbol{a}},\boldsymbol{b}\rangle|c(\bar{\boldsymbol{y}},\boldsymbol{z}).P;\Sigma,\boldsymbol{b}:T,c:S\rightharpoonup T]\!])\\
&= Tr^{S\rightharpoonup T}_{\Gamma\odot S,\Sigma\odot T}((\mathbf{symm}_{\Gamma,S\odot S\rightharpoonup T}); (\mathbf{symm}_{S,S\rightharpoonup T} \odot \mathrm{id}_\Gamma);\\
&\quad ([\![\bar{\boldsymbol{a}}:S,\bar{c}:S\rightharpoonup T \vdash \bar{c}\langle\bar{\boldsymbol{a}},\boldsymbol{b}\rangle;\boldsymbol{b}:T]\!] \odot [\![\Gamma \vdash c(\bar{\boldsymbol{y}},\boldsymbol{z}).P;\Sigma,c:S\rightharpoonup T]\!]);\\
&\quad (\mathbf{symm}_{T,\Sigma} \odot \mathrm{id}_{S\rightharpoonup T}))\\
&= ([\![\Gamma \vdash c(\bar{\boldsymbol{y}},\boldsymbol{z}).P;\Sigma,c:S\rightharpoonup T]\!] \odot \mathrm{id}_S); (\mathrm{id}_\Sigma \odot [\![\bar{\boldsymbol{a}}:S,\bar{c}:S\rightharpoonup T \vdash \bar{c}\langle\bar{\boldsymbol{a}},\boldsymbol{b}\rangle;\boldsymbol{b}:T]\!])\\
&= ((!\Lambda([\![\Gamma,\bar{\boldsymbol{y}}:S \vdash P;\Sigma,\boldsymbol{z}:T]\!]); \varrho_{S,\Sigma,T}; (\mathrm{id}_\Sigma \odot der_{S\rightharpoonup T})) \odot \mathrm{id}_S); (\mathrm{id}_\Sigma \odot \mathbf{app}_{S,T})
\end{aligned}
$$

(by a long but straightforward rewriting using the axioms of $Tr$ and **symm**)

$$\begin{aligned}
&= (!\Lambda([\![\Gamma, \bar{\boldsymbol{y}} : S \vdash P; \Sigma, \boldsymbol{z} : T]\!]) \odot \mathrm{id}_S); (\varrho_{S,\Sigma,T} \odot \mathrm{id}_S); (\mathrm{id}_\Sigma \odot \mathbf{app}_{S,T}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ((der \odot \mathrm{id}); \mathbf{app} = \mathbf{app}) \\
&= (!\Lambda([\![\Gamma, \bar{\boldsymbol{y}} : S \vdash P; \Sigma, \boldsymbol{z} : T]\!]) \odot \mathrm{id}_S); \mathbf{app}_{S,\Sigma\odot T} \qquad \text{(property of } \varrho) \\
&= [\![\Gamma, \bar{\boldsymbol{y}} : S \vdash P; \Sigma, \boldsymbol{z} : T]\!] \qquad \text{(universality of } \mathbf{app}) \\
&= [\![\Gamma, \bar{\boldsymbol{a}} : S \vdash P[\boldsymbol{b}/\boldsymbol{z}, \bar{\boldsymbol{a}}/\bar{\boldsymbol{y}}]; \Sigma, \boldsymbol{b} : T]\!]
\end{aligned}$$

Thus we have

$$[\![\nu x.(P[\boldsymbol{b}/\boldsymbol{z}, \bar{\boldsymbol{a}}/\bar{\boldsymbol{y}}]|Q)]\!] = [\![\nu x.\nu c.(\bar{c}\langle \bar{\boldsymbol{a}}, \boldsymbol{b}\rangle | c(\bar{\boldsymbol{y}}, \boldsymbol{z}).P|Q)]\!] \subseteq [\![\nu x.(\bar{x}\langle \bar{\boldsymbol{a}}, \boldsymbol{b}\rangle | x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P|Q)]\!].$$

□

### E.3 Labelled transition semantics

We relate our game model with a labelled transition system where only bound name is passed. This relation is used to show the correspondence between the intersection type system and the behaviour of a relationally describable processes. The proof is guided by Laird's work on investigating the connection between justified sequences and the labelled transition system [24].

The labelled transition system is defined as follows.

$$\frac{}{x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P \xrightarrow{x\langle \bar{\boldsymbol{k}}, \boldsymbol{l}\rangle} P[\bar{\boldsymbol{k}}/\bar{\boldsymbol{y}}, \boldsymbol{l}/\boldsymbol{z}]} \qquad \frac{}{\bar{x}\langle \bar{\boldsymbol{y}}, \boldsymbol{z}\rangle \xrightarrow{\bar{x}\langle \boldsymbol{k}, \bar{\boldsymbol{l}}\rangle} \boldsymbol{k} \to \bar{\boldsymbol{y}} | \boldsymbol{z} \to \bar{\boldsymbol{l}}}$$

$$\frac{P \xrightarrow{\alpha} P'}{P|Q \xrightarrow{\alpha} P'|Q} \qquad \frac{P \xrightarrow{\alpha} P'}{\nu(\bar{x}, y).P \xrightarrow{\alpha} \nu(\bar{x}, y).P'}$$

$$\frac{P \xrightarrow{\alpha} Q \qquad P \equiv P'}{P' \xrightarrow{\alpha} Q} \qquad \frac{P \xrightarrow{x\langle \bar{\boldsymbol{k}}, \boldsymbol{l}\rangle} P' \qquad Q \xrightarrow{\bar{x}\langle \boldsymbol{k}, \bar{\boldsymbol{l}}\rangle} Q'}{\nu(\bar{x}, x).(P|Q) \xrightarrow{\tau} \nu(\bar{x}, x).\nu(\bar{\boldsymbol{k}}, \boldsymbol{k}).\nu(\bar{\boldsymbol{l}}, \boldsymbol{l}).(P'|Q')}$$

Actions are either in the form of $\tau$, $x\langle \bar{\boldsymbol{k}}, \boldsymbol{l}\rangle$ and $\bar{x}\langle \boldsymbol{k}, \bar{\boldsymbol{l}}\rangle$, each of them corresponding to silent, input and output action, respectively. Each of the rules above is equipped with an implicit side condition that $P \xrightarrow{\alpha} P'$ only if $\mathbf{subj}(\alpha) \in \mathbf{fn}(P)$ and $(\mathbf{obj}(\alpha) \cup \mathbf{obj}(\bar{\alpha})) \cap \mathbf{fn}(P) = \emptyset$, where $\mathbf{subj}(\alpha)$ and $\mathbf{obj}(\alpha)$ is defined as the subject name of $\alpha$ and the set of object names of $\alpha$ respectively.

The following lemma relates our semantics to the labelled transition system. Here $\mathbf{trace}(P)$ is the $\tau$-free trace of the process $P$ and $\phi$ is a map from the set of traces to the set of justified sequence. (See [24] for the definition.) We define $\overline{\sigma}$ as the closure of (a set of interleaving plays) $\sigma$ by Condition (L1) to (L4) that is used to fill the gap between a trace of a process and a play in the interpretation of the process.

**Lemma 22.** $|[\![P]\!]| = \overline{\phi(\mathbf{trace}(P))}$.

*Proof.* By Theorem 3 and the fact that $[\![P]\!]_L = \overline{\phi(\mathbf{trace}(P))}$ [24], where $[\![P]\!]_L$ is the interpretation of $P$ in the interleaving semantics. □

## F Supplementary Materials for Section 5

### F.1 Time-forgetting map

The time-forgetting map forgets the sequential structure of a play. In this context, it should forget the causality relation $\rightsquigarrow$. We write $J_A^-$ for the set of all justified graphs without causality, i.e.,

$$J_A^- := \{(V, l, \curvearrowright) \mid (V, l, \curvearrowright, \emptyset) \in J_A\}.$$

**Definition 31 (Time-forgetting map).** *Given a play $s = (V, l, \curvearrowright, \rightsquigarrow)$ over $(A, B)$ and $X \in \{A, B\}$, we define $\mathcal{F}_X(s) = (V_X, l_X, \underset{X}{\curvearrowright})$ by*

$$\begin{aligned} V_X &:= \{v \in V \mid l(v) \in \mathcal{M}_X\} \\ l_X(v) &:= l(v) \\ \underset{X}{\curvearrowright} &:= (\curvearrowright) \cap (V_X \times V_X). \end{aligned}$$

*We define the same operation for an interaction graph. For a strategy $\sigma : A \to B$ in $\mathcal{P}$, we define $\mathcal{F}(\sigma) := \{(\mathcal{F}_A(s), \mathcal{F}_B(s)) \mid s \in \sigma\}$. So $\mathcal{F}(\sigma) \subseteq J_A^- \times J_B^-$ is a relation.*

The time-forgetting map $\mathcal{F}$ is a lax functor (but not a functor).

**Proposition 4.** *$\mathcal{F}(\tau \circ \sigma) \subseteq \mathcal{F}(\tau) \circ \mathcal{F}(\sigma)$, where the composition in the right-hand-side is that of relations.*

*Proof.* Assume that $s \in \tau \circ \sigma$. Then there exists an interaction graph $u \in \mathrm{Int}(A, B, C)$ such that $u{\upharpoonright_{A,B}} \in \sigma$, $u{\upharpoonright_{B,C}} \in \tau$ and $u{\upharpoonright_{A,C}} = s$. It is not difficult to see the following equation for $X \in \{(A, B), (B, C), (A, C)\}$ and $Y \in \{A, B, C\}$ such that $Y$ appears in $X$:

$$\mathcal{F}_Y(u) = \mathcal{F}_Y(u{\upharpoonright_X}).$$

So we have $(\mathcal{F}_A(u), \mathcal{F}_C(u)) \in \mathcal{F}(\tau) \circ \mathcal{F}(\sigma)$. By the above equation, we have $\mathcal{F}(s) = (\mathcal{F}_A(u), \mathcal{F}_C(u))$ as desired. $\square$

### F.2 Relationally describable processes

A pair $(s, t)$ of plays $s \in P_{A,B}$ and $t \in P_{B,C}$ is *composable* if $\mathcal{R}_B(s) = \mathcal{R}_B(t)$. A composable pair $(s, t)$ induces a "justified graph" $u = (V, l, \curvearrowright, \rightsquigarrow)$ given by

$$\begin{aligned} V &:= V_s \cup V_t \\ l(v) &:= \begin{cases} l_s(v) & (\text{if } v \in V_s) \\ l_t(v) & (\text{otherwise}) \end{cases} \\ \curvearrowright &:= (\underset{s}{\curvearrowright}) \cup (\underset{t}{\curvearrowright}) \\ \rightsquigarrow &:= (\underset{s}{\rightsquigarrow}) \cup (\underset{t}{\rightsquigarrow}) \end{aligned}$$

where we assume that $V_s \cap V_t = \{v \in V_s \mid l_s(v) \in \mathcal{M}_B\} = \{v \in V_t \mid l_t(v) \in \mathcal{M}_B\}$. It is not difficult to see that $u$ satisfies all the requirements for an interaction graph *except for acyclicity*. The composable pair $(s,t)$ *has a cycle* if the induced justified graph is cyclic. Otherwise it is *cycle-free*.

Let $\sigma : A \to B$ and $\tau : B \to C$ be strategies. They are *cycle-free* if any composable pair $(s,t) \in \sigma \times \tau$ is cycle-free. In this case, we also say that the composition of $\sigma$ and $\tau$ is cycle-free.

For cycle-free compositions, the time-forgetting map $\mathcal{F}$ behaves like a functor.

**Lemma 23.** *Let $\sigma : A \to B$ and $\tau : B \to C$ be strategies. If $(\sigma, \tau)$ is cycle-free, then $\mathcal{F}(\tau \circ \sigma) = \mathcal{F}(\tau) \circ \mathcal{F}(\sigma)$.*

*Proof.* Let $x \in J_A$ and $z \in J_C$ and assume $(x,z) \in \mathcal{F}(\tau) \circ \mathcal{F}(\sigma)$. Then there exists $y \in J_B$ such that $(x,y) \in \mathcal{F}(\sigma)$ and $(y,z) \in \mathcal{F}(\tau)$. Hence there exists $s \in \sigma$ such that $\mathcal{F}_A(s) = x$ and $\mathcal{F}_B(s) = y$. Similarly there exists $t \in \sigma$ such that $\mathcal{F}_B(t) = y$ and $\mathcal{F}_C(t) = z$. So $(s,t)$ is a composable pair. Since $(\sigma, \tau)$ is cycle-free, the induced justified graph $u$ is acyclic, and thus a (genuine) interaction graph. Hence $u{\restriction}_{A,C} \in (\tau \circ \sigma)$. So

$$(x,z) = (\mathcal{F}_A(s), \mathcal{F}_C(t)) = (\mathcal{F}_A(u), \mathcal{F}_C(u)) = (\mathcal{F}_A(u{\restriction}_{A,C}), \mathcal{F}_C(u{\restriction}_{A,C})) \in \mathcal{F}(\tau \circ \sigma).$$

The other direction is proved in Proposition 4. $\square$

A process $P$ is *relationally describable* if all the compositions appearing in the interpretation of $P$ are cycle-free. If $P$ is relationally describable, then one do not need the causality relation to compute $\mathcal{F}([\![P]\!])$.

**Intersection type system** The typing rules are listed in Figure 14.

A type $\varphi$ (resp. an intersection type $\xi$) is a *refinement of simple type $S$* if $\varphi :: S$ (resp. $\xi :: S$) is derivable by the following rules:

$$\frac{\xi_1 :: S_1 \quad \dots \quad \xi_n :: S_n \quad \zeta_1 :: T_1 \quad \dots \quad \zeta_k :: T_k}{\mathbf{ch}[\xi_1 \dots \xi_n, \zeta_1 \dots \zeta_k] :: \mathbf{ch}[S_1 \dots S_n, T_1 \dots T_k]} \qquad \frac{\forall i \le n.\ \varphi_i :: S}{\langle \varphi_1, \dots, \varphi_n \rangle :: S}$$

The base case is the empty intersection type $\langle\rangle :: S$ for every $S$. A *refinement of an input type environment $\Sigma = x_1 : S_1, \dots, x_n : S_n$*, ranged over by $\Theta$, is of the form $x_1 : \xi_1, \dots, x_n : \xi_n$ with $\xi_i :: S_i$ for every $i$. We write $\Theta :: \Sigma$ if $\Theta$ is a refinement of $\Sigma$. For an output type environment $\Gamma$, the relation $\Xi :: \Gamma$ is defined by a similarly way.

A derivation $\Xi \vdash P;\ \Theta$ is a *refinement* of $\Gamma \vdash P;\ \Sigma$ if, for every type binding $x : \xi$ in $\Xi \vdash P;\ \Theta$, we have $\xi :: T$ where $T$ is the type for the name $x$ in $\Gamma \vdash P;\ \Sigma$.

**Lemma 24.** *Let $\Gamma \vdash P;\ \Sigma$ be a process and $\Xi :: \Gamma$ and $\Theta :: \Sigma$ be refinements of the type environments. If $\Xi \vdash P;\ \Theta$, then there exists a derivation of the same judgement that is a refinement of $\Gamma \vdash P;\ \Sigma$.*

There are bijections between $\{\xi \mid \xi :: T\}$ and $J^{-}_{[\![T]\!]}$ and between $\{\Xi \mid \Xi :: \Gamma\}$ and $J^{-}_{[\![\Gamma]\!]}$. We write $(\!|\Xi|\!)$ for the latter map.

The above intersection type system is related to the type-forgetting map as follows.

**Lemma 25.** *Let $\Gamma \vdash P; \Sigma$ be a relationally describable process. Then $\mathcal{F}([\![\Gamma \vdash P; \Sigma]\!]) = \{((\!|\Xi|\!), (\!|\Theta|\!)) \mid \Xi \vdash P;\ \Theta\}$, where $\Xi \vdash P;\ \Theta$ in the right-hand-side of the equation is restricted to refinements of $\Gamma \vdash P; \Sigma$.*

*Proof.* By induction on the structure of $P$. We use the fact that we can simply ignore the causal relation since $P$ is relationally describable. $\square$

**Proof of Theorem 7** A consequence of Lemmas 22 and 25.

$$\frac{\varXi, \bar{x}:\xi, \bar{y}:\zeta \vdash P;\ \varTheta}{\varXi, \bar{z}:\xi \wedge \zeta \vdash P\{\bar{z}/\bar{x}, \bar{z}/\bar{y}\};\ \varTheta}$$

$$\frac{\varXi \vdash P;\ \varTheta, x:\xi, y:\zeta}{\varXi \vdash P\{z/x, z/y\};\ \varTheta, z:\xi \wedge \zeta}$$

$$\frac{\varXi, \bar{x}:\xi, \bar{y}:\zeta, \varXi' \vdash P;\ \varTheta}{\varXi, \bar{y}:\zeta, \bar{x}:\xi, \varXi' \vdash P;\ \varTheta}$$

$$\frac{\varXi \vdash P;\ \varTheta, x:\xi, y:\zeta, \varTheta'}{\varXi \vdash P;\ \varTheta, y:\zeta, x:\xi, \varTheta'}$$

$$\frac{}{\emptyset \vdash \mathbf{0};\ \emptyset}$$

$$\frac{\varXi \vdash P;\ \varTheta \qquad \varXi' \vdash P';\ \varTheta'}{\varXi, \varXi' \vdash P|P';\ \varTheta, \varTheta'}$$

$$\frac{}{\bar{x}:\mathbf{ch}[\boldsymbol{\xi}, \boldsymbol{\zeta}], \bar{\boldsymbol{y}}:\boldsymbol{\xi} \vdash \bar{\bar{x}}\langle \bar{\boldsymbol{y}}, \boldsymbol{z}\rangle;\ \varTheta, \boldsymbol{z}:\boldsymbol{\zeta}}$$

$$\frac{\varXi, \bar{\boldsymbol{y}}:\boldsymbol{\xi} \vdash x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P;\ \varTheta, \boldsymbol{z}:\boldsymbol{\zeta}}{\varXi \vdash x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P;\ \varTheta, x:\mathbf{ch}[\boldsymbol{\xi}, \boldsymbol{\zeta}]}$$

$$\frac{\forall i \leq I.\ \varXi_i \vdash x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P;\ \varTheta_i}{\bigwedge_{i \in I} \varXi_i \vdash !x(\bar{\boldsymbol{y}}, \boldsymbol{z}).P;\ \bigwedge_{i \in I} \varTheta_i}$$

$$\frac{\varXi, \bar{x}:\xi \vdash P;\ \varTheta, y:\xi}{\varXi \vdash \nu(\bar{x}, y).P;\ \varTheta}$$

Fig. 14: Typing rules for the intersection type system